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This sixth unit in the SMSG junior high mathematics series is a student text covering the following topics: permutations and selections; probability; similar triangles and variation; non-metric geometry; volumes and surface areas; the sphere; and unsolved problems in mathematics. (DT)

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STUDENT'S TEXT

UNIT NO. 6

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MATHEMATICS FOR JUNIOR HIGH SCHOOL VOLUME 2

PART II

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SCHOOL MATHEMATICS STUDY GROUP

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School Mathematics Study Group

Mathematics for Junior High School, Volume 2

Unit 6

Mathematics for Junior High School, Volume 2

Student's Text, Part II

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Chapter 7

PERMUTATIONS AND SELECTIONS

7-1. The Pascal Triangle

Five students form a club. We shall call them by their initials A, B, C, D, E. Naturally, the first order of business in the club is to choose a refreshment committee. It is agreed that the committee should have three members. How many possibilities for the membership on the committee do you think there are?

One possibility would be a committee consisting of B, C, and E. We might abbreviate this possible committee by the symbol {B,C,E}.

Class Exercises 7-1a

1. Another possible membership list consists of A, D, and E. Write the abbreviated symbol for this case.
2. Does the committee {B,E,A} have the same members as the committee {E,A,B}?
3. Give two other symbols, each of which names the committee mentioned in Problem 2.
4. Make a list of all the possible committees of three members.
5. How many committees are in your list?
6. Of how many of these committees is D a member?
7. Compare the number of committees of which B is a member and the number of those which include D.

(Answer the following questions without doing any more counting.)

8. How many of the committees do not include A?

9. What is the ratio of the number of committees including E to the number of possible committees? (Did you answer this question without further counting?)

10. How many committees have both A and C as members? We may easily answer this question without looking at our list of all the possible committees. We observe that since a committee {A,C,?} has two members specified, then there is only one vacancy to be filled. How many possible choices are there for the third member? Thus three of the ten possible committees include both C and A.

11. What is the ratio of the number of possible committees including both B and E to the number of committees including B?

Whenever three of the five students are chosen for a special purpose, such as membership on a committee, then the remaining two have also been chosen--chosen, in the sense of not serving on this particular committee. In other words, the selection of a committee, in effect, separates the club members into two sets. One method for selecting the membership of a committee is to decide which club members will not serve. For example, if it is decided that a committee should not include C and D, then we know that the committee is {A,B,E}.

12. Name the committee determined by the condition that A and E have been chosen to be non-members.

13. Which two students are picked as being non-members in {E,B,C}? The selection of a committee of three members also means a choice of another committee with two members, namely the other two of the five club members. For example, the selection of {E,B,C} determines the two-member set or committee {A,D}.

14. Since there are ten possible committees with three members each, how many possible committees with two members each are there?

15. Since six of the possible three-member committees include B, how many of the possible two-member committees exclude B?

16. How many of the possible two-member committees include C?

17. Find the answer to Problem 16, using the method of filling the vacancy in {C,?}.

Exercises 7-1a

1. From the club {A,B,C,D,E}, one possible committee with four members is {A,B,D,E}.
   (a) Make a list of all the possible committees, each with four members.
   (b) How many in the club are excluded each time a committee with four members is formed?
   (c) Make a list of all the possible committees with one member each.
   (d) What relationship is there between the number of possible committees with four members each and the number of possible committees with one member each?
   (e) How many committees are there with all five as members?

2. A club has four members whom we may call K, L, M, N.
   (a) How many possible committees in this club could have four members?
   (b) How many possible committees could be formed with one member each?
   (c) Name each of the possible one-member committees.
   (d) To each one-member committee there corresponds, in a natural way, a committee with three members. What is that natural way?
   (e) Use parts (c) and (d) to make a list of the possible committees with three members each.

(f) Make a list of the possible committees with two members each.

3. Make a list of all the possible committees, and note how many committees there are of each size in a club with three members. (Call the club members P, Q, and R.)

4. Do as directed in Problem 3 for a club with two members. Name the members U and V.

5. Do as directed in Problem 3 for a club with just one member.

6. A family would enjoy each of four vacation spots. It is decided to choose two of the four and spend part of the vacation time at each of the two. How many possible choices are there for the pair of vacation places?

7. The refrigerator holds two cartons of ice cream. The dairy has five flavors, and the family always likes to buy two different flavors. How many times can the family go to the dairy and bring home a different pair of flavors?

Let us make a table showing the number of possible committees with a given number of members from a club with five members. This table will summarize several of the results we have obtained in previous problems. In a club with five members, there are 5 possible committees with one member, 10 committees with two members, 10 committees with three members, 5 committees with four members, 1 committee with five members. The selection of a committee which includes all five club members (sometimes referred to as the "committee of the whole") means that there are zero club members not serving. Thus we may balance our table by saying that there is one possible committee with zero members. (You may wish to compare this agreement with the remark that there is just one empty set.)

If we arrange our data according to increasing size of committees, we have the following sequence:

1 5 10 10 5 1

These six numbers tell us how many possible committees of various sizes can be chosen from a club membership of five.

The same type of data, for a club membership of four, is the following:

1 4 6 4 1

Be sure that you understand the significance of each of these five entries.

We now have two of the rows in the table we are constructing.

Class Exercises 7-1b

1. In particular, what does the last 1 in the data 1, 4, 6, 4, 1 mean?
2. What does the first 1 mean?
3. What is the corresponding listing for a club membership of three?
4. How can you interpret the data

   1 2 1 ?

5. What data of this type do we have for a club with only one member?

   Let us collect into a table the data for the various clubs. Each row of the table below shows the information for a club of a certain size.

```
              1   1
            1   2   1
          1   3   3   1
        1   4   6   4   1
      1   5   10  10  5   1
```

Let us examine again the entry in the table telling how many possible committees of three members each can be named from the club {A,B,C,D,E}. In the table the entry is which of the 10's?

A committee of three may include E or it may not. We will study these two cases in more detail.

6. How many possible committees with three members include E?
7. A committee including E is of the type {E,?,?}. How many vacancies appear? From how many members can these vacancies be filled?
8. In view of Problem 7, compare the answer to Problem 6 with the number of possible two-member committees in a club of four members.
9. How many possible committees with three members exclude E?
10. A committee excluding E is of the type {?,?,?} where no blank may be filled with E. How many vacancies appear? From how many possibilities can these vacancies be filled?
11. In view of Problem 10, the answer to Problem 9 is the same as the number of committees with ____(how many?)____ members from a club of ____(how many?)____ members.

By encircling we show in the table below the three entries we have been studying.

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

The entry 10 is the sum of the two numbers, 6 and 4, nearest it on the preceding line.

The table we have been studying is a part of the array known as the Pascal triangle. (The French mathematician, Pascal, seventeenth century, contributed to geometry and the theory of probability.) The table would resemble even more an equilateral triangle if we supplied a vertex at the top; this is sometimes done, but we shall not be concerned with it. In our version of the Pascal triangle, the first, second, third, fourth, and fifth rows show the numbers of possible committees from a club of one, two, three, four, and five members, respectively. Copy the "triangle" and add the row corresponding to a club with six members.

Exercises 7-1b

1. Check that (except for the ones) every entry in the table is the sum of the two numbers nearest it on the preceding line.
2. (a) What does the 6 in the fourth row mean?

   (b) What does the first 3 in the third row mean?
3. (a) What does the second 15 mean in the sixth row?

   (b) What does the second 10 mean in the fifth row?
4. (a) Which entry indicates the number of possible committees with two members formed from a club of six members?

   (b) Which entry indicates the number of possible committees with one member formed from a club of five members?
5. A club has six members, which we denote as A, B, C, D, E, F.

   (a) Some of the possible committees with two members are {A,B}, {A,C}, {C,E}. Make a list of all fifteen of these committees. (Write your list down a page, using fifteen rows.)

   (b) On the right-hand side of your answer to (a), make a list of all the possible committees with four members. Specifically, for each committee in the list for (a), write beside it the committee whose four members are excluded from the committee with two members. As an example, one line on your answer sheet will be:

   {A,C}   {B,D,E,F}

   (c) After you have written your list of committees with four members each, how can you obtain the number of these possible committees by adding two numbers obtained from the fifth row of the Pascal triangle?

(d) Make a complete list of all possible committees with three members each.

(e) Does the number of committees listed in (d) agree with a number obtained from the fifth row of the Pascal triangle?

6. Find the seventh row of the Pascal triangle.

7. Find the eighth row of the Pascal triangle.

8. What are the first two entries (on the left) in the twenty-third row of the Pascal triangle?

9. What are the last two entries (on the right) in the fifty-seventh row of the Pascal triangle?

7-2. Permutations

Suppose that the club whose five members are A, B, C, D, and E chooses an executive committee to conduct the business. The executive committee has three members and is composed of B, D, and E. These three members, in a meeting of the committee, decide that they should assign responsibilities. One should be chairman, another be secretary, and the third be treasurer for the club. In how many ways do you believe these jobs can be given to the three?

Class Exercises 7-2a

1. If D is chosen chairman, in how many different ways can the other two jobs be distributed between B and E?

2. List each of these ways in detail, by telling which job each one would have.

3. If E is chosen chairman, in how many different ways can the other two be given jobs?

4. In how many different ways can the three offices be assigned to the three if B is chairman?

5. In how many different ways can the three offices be assigned to the three members of the executive committee?

Exercises 7-2a

1. A club has eight members whose initials are A, B, C, D, E, F, G, H. An executive committee (A,F,H) distributes its jobs among its members. One possible way is:

   Chairman A, Secretary H, Treasurer F.

   (a) Make a list of all possible ways of assigning these three jobs to the three members of the committee so that each person has a job.

   (b) How many ways are there?

2. Four boys--Paul, Ron, Sam, and Ted--will participate, one after another in a relay race.

   (a) One possible order of running is as follows: First, Paul; second, Sam; third, Ron; fourth, Ted. Make a list of all the possible orders of performance. Note: One way to make this list is to fix attention on the position and tabulate how the boys can be fitted in (use P, R, S, and T to represent the boys). Such a table might begin like this:

Runner 1	Runner 2	Runner 3	Runner 4	1	2	3	4	1	2	3	4	1	2	3	4
P	R	S	T	R	P	S	T	S	P	R	T	T	P	R	S
P	R	T	S	R	P	T	S	S	.	.	.	T	.	.	.
P	S	R	T	R	.	.	.	S	.	.	.	T	.	.	.
P	S	T	R	R	.	.	.	S	.	.	.	T	.	.	.
P	T	R	S	R	.	.	.	S	.	.	.	T	.	.	.
P	T	S	R	R	.	.	.	S	.	.	.	T	.	.	.

Complete the listing. Save it for further use.

(b) How many different teams are in your list?

3. Assume that a group of people are asked in a poll to express their preferences concerning potatoes. The possible choices are baked, mashed, and french-fried potatoes. They are to indicate which they like best, next best, and least. How many different orderings of preferences are possible?

4. Three different presents are given to three children. In how many different ways can the gifts be distributed among the children?

5. Four horses are to be assigned positions at the post in a race. In how many ways is it possible to distribute the horses among the first, second, third, and fourth positions? If you prefer, you may use the answer to another problem in this set of exercises rather than making a new list.

6. A salesman works five days during the week. He has customers in five cities. He spends one day each week in each city. Clayville is only six miles from his home and he goes there each Monday. Since he does not enjoy routine in his traveling, he likes to match the other weekdays with the four

remaining cities in as many ways as possible. How many weeks can he work without being obliged to repeat any route for a week?

7. A stenographer has four envelopes addressed to Adams, Brown, Clark, and Davis, respectively. She has four letters written to these four men. She puts one letter in each envelope. In how many ways might she do this so that one or more of the letters is placed in the wrong envelope? Is it possible to place just one letter in the wrong envelope?

There is an obvious difference between the problems you have just been working and the problems of Section 7-1. In Section 7-1 the order in which you named individuals did not matter. For example, the committee {A,B,C} was the same as committee {C,B,A}. There, we were interested only in the set containing the three elements A, B, and C.

In the last group of class exercises, we were forming executive committees of the individuals B, D, and E. Let us agree that, when we name such a committee, the first-named will be chairman, the second will be secretary, and the third treasurer. Thus {B,D,E} would represent B as chairman, D as secretary and E as treasurer. For example, the executive committee {E,B,D} would be different from the executive committee {D,E,B}. In the relay race problem, the relay team PTRS would be different from the team RTPS, because the order in which the boys run is different. In Problem 3 of the last set, the preference listing {mashed, french-fried, baked} is different from the listing {french-fried, mashed, baked} because of the order in which the items are listed. Such problems in which the order is important are called permutation problems.

Different arrangements (or orderings) of the objects or persons are of interest in a permutation problem. We would say that PTRS and RTPS are two different permutations (or arrangements) of P, R, S, and T. So, in the relay race problem, we want to count the *number of permutations* of four things, namely P, R, S, and T. In the preference-poll problem, we needed to count the number of permutations of three things; namely, three ways of cooking potatoes.

> *Definition*. *A permutation of a set of elements is an arrangement of the elements of the set in some order*.

How to Count Permutations.

Up to this time you have been counting the number of permutations merely by listing them. You had to be careful to list them in an orderly fashion and not to miss some permutations. A faster and more efficient method of counting is needed, especially if a large number of objects is involved.

Suppose you want to indicate your preference for three flavors of ice cream--vanilla, strawberry, and chocolate (name these by letters V, S, and C). You want to designate first, second, and third preferences. Your possible listings are, by columns:

1	2	3
V S C	S V C	C S V
V C S	S C V	C V S

Note that V is first preference in Column 1; S is first preference in Column 2; C is first preference in Column 3. This indicates that you could choose your first preference in any one of three ways. Suppose now that you have chosen first preference as V. There were two ways of choosing the second preference, either S or C. (You can see this in Column 1.) If your first preference had been S, how many choices were there for second preference? If your first preference had been C, how many were there for second choice? For each possible first preference there were two possible second preferences. Since the first preference could be chosen in any of 3 ways, and for *each* of these, the second could be chosen in 2 ways, the total number of choices for the first two preferences is what number? It is hoped that you said $3 \cdot 2$ ways. There remained then only one choice for the third preference. Hence the total number of choices for all 3 preferences is $3 \cdot 2 \cdot 1$.

Another way of thinking about this problem is to use boxes to indicate the three preferences.

$$\boxed{3}\boxed{2}\boxed{1}$$

For your first preference you have 3 possible choices, which you may indicate by a 3 in the first box. Once this first preference has been given, you have only two possible choices for second preference. This is indicated by placing a 2 in the second box. Now, having chosen your first preference and also your second preference, there is only one possible third preference, which you indicated by a 1 in the third box. Thus, the total number of preferences is $3 \cdot 2 \cdot 1$. Again, we observe that, for each choice for first position, there are two choices for second position.

As another illustration of this box device, let us look at the possible different running orders for the relay team of P, R, S, and T discussed in Problem 2, Exercises 7-2a. For first runner, we may choose any one of the four boys. We indicate this by a 4 in the first box of the diagram below:

$$\boxed{4}\boxed{3}\boxed{2}\boxed{1}$$

Having chosen the first runner, we may choose any of the 3 remaining boys to run in second position. With any specified choice for the first two positions, we have two possible choices for the third position. Finally, having chosen three boys, there remains only 1 choice for the fourth position on the team. Hence, the total number of possible running orders is $4 \cdot 3 \cdot 2 \cdot 1 = 24$. These are the 24 orderings of PRST which you enumerated in the last paragraph.

Exercises 7-2b

1. Two-digit numerals are to be formed using the digits 6, 7, and 8. No digit is to be used more than once (that is, numerals like 77 are not permitted here).

   (a) How many choices are there for the first digit?

   (b) The first digit having been chosen, how many choices are there for the second digit?

   (c) How many two-digit numerals of the type permitted can be written? (Leave the answer as an indicated product.)

In the following problems leave all answers as indicated products ($9 \cdot 8$ should not be written as 72).

2. Use digits 1, 2, 3, 4, 5, 6, 7, 8, and 9 and form two-digit numerals as in Problem 1. How many such can be formed?

3. How many different two-letter "words" can be formed using the letters of our alphabet? No letter is to be used more than once. (The "word" formed need not make sense--the two-letter arrangement tg is a "word" in this sense!)

4. Use the digits 6, 7, 8, and 9, not permitting the repetition of any digit (as in Problem 1). How many four-digit numerals can be formed?

5. Use digits 1, 2, 3, 4, 5, 6, 7, 8, and 9; repetition of a digit is not permitted.

   (a) How many three-digit numerals can be formed?

   (b) How many four-digit numerals can be formed?

   (c) How many six-digit numerals can be formed?

6. Four persons enter a room which contains 15 chairs arranged in a row. In how many different ways could the persons be seated in this row?

7. Suppose there are n chairs placed in a row. Two persons are to be seated.

   (a) How many choices does the first person have?

(b) After the first person is seated, how many different choices remain for the second person?

(c) Is your answer to part (b) the same for each chair the first person may choose? Why?

(d) How many different pairs of chairs can the two people choose?

(e) Find the number of ways three persons can choose chairs from the n chairs placed in a row.

(f) Find the number of ways four persons can choose chairs from the n chairs placed in a row.

In Problem 1 above you found the number of permutations (or orderings) of 3 different things arranged two at a time. We will use the symbol $P_{3,2}$ for this number. In Problem 3, you were asked to find $P_{26,2}$, the number of permutations of 26 different things arranged 2 at a time. Using this notation in Problem 7(d) we wished to find $P_{n,2}$; in Problem 7(e) we wanted to find $P_{n,3}$, the number of permutations of n different things arranged 3 at a time.

In general, we say:

$P_{n,r}$ = the number of permutations of
n different things arranged
r at a time.

According to your results in the preceding problems:

$$P_{4,2} = 4 \cdot 3 \qquad P_{9,2} = 9 \cdot 8$$

$$P_{4,3} = 4 \cdot 3 \cdot 2 \qquad P_{26,2} = 26 \cdot 25$$

$$P_{4,4} = 4 \cdot 3 \cdot 2 \cdot 1 \qquad P_{15,4} = 15 \cdot 14 \cdot 13 \cdot 12$$

$$P_{n,2} = n(n-1) \qquad P_{n,3} = n(n-1)(n-2).$$

The symbol $P_{n,r}$ makes sense only when n and r are counting numbers and $r \le n$.

There is a special case of $P_{n,r}$ which is of considerable importance. In Problem 4 you were finding "the number of permutations of 4 things arranged 4 at a time," or, in abbreviated form, $P_{4,4}$. The answer was $P_{4,4} = 4 \cdot 3 \cdot 2 \cdot 1$. This number is the product of all the counting numbers in succession from 1 to 4. Similarly, $P_{5,5} = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ is the product of all counting numbers from 1 to 5. Such products using successive counting numbers as factors occur frequently in mathematics and we have a special symbol for them. We write $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ and we read $5!$ as "five factorial." Similarly, "four factorial" is $4! = 4 \cdot 3 \cdot 2 \cdot 1$.

In general, n factorial (written $n!$) means the product of all counting numbers in succession from 1 to n. Thus,

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ \ldots \ \cdot 3 \cdot 2 \cdot 1.$$

Note that it is equally correct to write:

$$4! = 1 \cdot 2 \cdot 3 \cdot 4$$

$$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \quad \text{and}$$

$$n! = 1 \cdot 2 \cdot 3 \cdot \ \ldots \ \cdot (n-1) \cdot (n).$$

In much of your work here it is probably more convenient to write $n!$ as $n \cdot (n-1) \cdot \ \ldots \ \cdot 2 \cdot 1$, but you may write $n! = 1 \cdot 2 \cdot 3 \cdot \ \ldots \ \cdot n$ if you wish.

You should check, on a separate piece of paper, that

$$1! = 1$$

$$2! = 2 \cdot 1 = 2$$

$$3! = 3 \cdot 2 \cdot 1 = 6$$

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5{,}040$$

$$8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40{,}320.$$

As you see, n factorial increases at a truly remarkable rate as n increases. Hence the exclamation point "!" is an appropriate symbol to use. (To express the same sentiment, British mathematicians sometimes read n! as "n admiration"!)

In our work on permutations we noted that:

$$P_{3,3} = \left\{\begin{array}{l}\text{the number of permutations of } 3\\ \text{different things arranged } 3 \text{ at}\\ \text{a time}\end{array}\right\} = 3!, \text{ and}$$

$$P_{4,4} = \left\{\begin{array}{l}\text{the number of permutations of } 4\\ \text{different things arranged } 4 \text{ at}\\ \text{a time}\end{array}\right\} = 4!\ .$$

By using arguments like those of the previous paragraphs you should be able to convince yourself of the truth of the following:

> If n is a counting number, the number of permutations of n different things arranged n at a time is n factorial. In symbols, we write
>
> $$P_{n,n} = n!$$

Exercises 7-2c

1. Express the following in product form:
   (a) 6! (b) 7! (c) 10! (d) 15!
2. Notice that $4! = 4\cdot3\cdot2\cdot1 = 4(3\cdot2\cdot1) = 4(3!)$. In a similar fashion, write each of the following factorials in terms of a second factorial.
   (a) 7! (b) 6! (c) 10! (d) 12!
3. Find the quotient of 14! divided by 13! (without performing any multiplications).
4. Show, without performing any multiplications, that 6! is the product of 6, 5, and (4!).
5. The factorial of 10 is the product of 10 and 9 and another factor. What is this third factor?
6. Show that 62! is the same as $62\cdot61\cdot(60!)$.

7. How many different batting orders are possible for a baseball team of nine players?

8. In a racing boat there are 8 seats, one behind another. In how many ways can the 8 members of a university crew take these seats?

9. How many permutations are there of the letters of the word "scholar"?

10. If one of the members of a baseball team always pitches, in how many different playing arrangements can the other team members be distributed among the other playing positions? There are only nine members available for the team.

A General Multiplication Property

In our thinking about arrangements and selections we have often made use of the following:

> Multiplication Property. *If an operation can be done in m ways and, after it has been performed in any one of these ways, a second operation can be performed in n ways, then the two successive operations can be performed in $m \times n$ ways.*

As a pleasant illustration of this property, think of the problem which faces you in choosing a sundae at a dairy or drug store. You have a choice of 3 flavors of ice cream (strawberry, vanilla, and chocolate). After you have chosen the flavor of ice cream, you may choose either of 2 toppings (marshmallow or nut). You may perform the first selection in 3 ways and then, after you have chosen any particular flavor, the second choice may be made in 2 ways. Thus the total number of different sundaes is $3 \cdot 2$ or 6. In words,

$$\left\{\begin{matrix}\text{The number}\\ \text{of different}\\ \text{sundaes}\end{matrix}\right\} = \left\{\begin{matrix}\text{the number}\\ \text{of different}\\ \text{flavors}\end{matrix}\right\} \times \left\{\begin{matrix}\text{the number}\\ \text{of different}\\ \text{toppings}\end{matrix}\right\}.$$

As a second illustration, we ask: How many possible license plates are there consisting of a letter followed by 2 digits? Do not allow zero as a first digit.

We think of the problem in terms of a box diagram.

26	9	10

The first position on the license plate can be filled in 26 possible ways, since we may use any one of the 26 letters of the alphabet. The second box may be filled in 9 ways, since we do not allow a zero in this position. In the third position we may use any one of the 10 digits. In all, then, there are $26 \cdot 9 \cdot 10 = 26 \cdot 90 = 2340$ different license plates possible.

Note that in the preceding example we have used the multiplication property for three successive operations. Indeed we often use this type of thinking for a number of successive choices.

Exercises 7-2d

1. A boy has seven shirts and four pairs of trousers. How many different combinations of a shirt and a pair of trousers can he choose?

2. A baseball team has five pitchers and three catchers. How many batteries (consisting of a pitcher and a catcher) are possible?

3. If the first two call letters of a television station must be KT, how many calls of four different letters are possible?

4. A disc jockey has 50 records in his collection. He wants to make a program of two different records. How many possible programs are there? (Count different orderings of the same records as different programs.)

5. A signalman has six flags. The emblems on the various flags are: a stripe, a dot, a triangle, a rectangle, a bar, and a circle. By showing two different flags, one after the other,

the signalman can send a signal. How many different signals are possible?

6. How many possible license plates are there consisting of one letter followed by 3 digits? (The first digit may not be zero. As in the illustration above, repetition of digits is allowed for the 2nd and 3rd digits.)

7. A set of five flags has one of each of the colors red, green, yellow, blue, and white as a signal. Three flags are to be hoisted, one above the other on the same mast. How many different signals are possible?

8. How many different license plates are possible using two letters followed by two digits? The first digit may not be zero.

9. How many license plates would be possible using 4 digits, the first of which may not be zero?

10. A student has 10 different books, 5 of which he wishes to arrange between book-ends on his desk. How many different arrangements are there?

A General Permutation Formula

Suppose we have a set of seven flags, each one a different color. How many different signals may we form from 3 flags, hoisted vertically on the same mast?

A signal thus means an arrangement of 3 of the 7 flags, or, as we have said, a permutation of 7 things arranged 3 at a time. We use the symbol $P_{7,3}$, you recall. The first flag may be selected from any one of the 7 possibilities. After it has been selected, there remain 6 possibilities for the second flag. After both these choices have been made, there remain 5 ways of selecting the third flag. By the multiplication property, the

7
6
5

total number of permutations is $7 \cdot 6 \cdot 5$, or

$$P_{7,3} = 7 \cdot 6 \cdot 5.$$

This way of thinking allows us to write a general formula for the number of permutations of n things taken r at a time. There are n possible choices for the first selection, then since there are $(n-1)$ objects left, there are $(n-1)$ possible choices for the second selection, $(n-2)$ for the third and so on. There will be r stages in this procedure, one for each of the objects being used in the permutation. Hence, there will be r factors in the final product. Accordingly,

$$P_{n,r} = n(n-1)(n-2) \ldots \text{ to } r \text{ factors, where } r \leq n.$$

As an illustration of this, we see that

$$P_{7,5} = \underbrace{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}_{5 \text{ factors}}, \quad \text{when } n = 7, \ r = 5$$

and $$P_{24,3} = \underbrace{24 \cdot 23 \cdot 22}_{3 \text{ factors}}, \quad \text{when } n = 24, \ r = 3.$$

Exercises 7-2e

1. Write the product $49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 \cdot 43$ using the form $P_{n,r}$.
2. (a) Write the number $P_{12,3}$ in factored form (but do not multiply).

   (b) Write the product of $P_{12,3}$ and $9!$ in expanded form (but do not multiply).

   (c) Is $P_{12,3} \cdot (9!) = P_{12,12}$? Why?
3. Express $P_{12,3}$ as the quotient of two numbers, each of which is a factorial. Hint: See Problem 2(c).
4. (a) Write the number $P_{20,4}$ in product form (but do not multiply).

   (b) Write the product of $P_{20,4}$ and $16!$ in product form (but do not multiply).

(c) What convenient name do we have for the product in (b)?

5. Express $P_{20,4}$ as the quotient of two numbers, each of which is a factorial.

6. Problems 2, 3, 4, and 5 suggest a way of expressing $P_{n,r}$ in terms of factorials. Write $P_{n,r}$ as a product as is done in the paragraph preceding these exercises.

   (a) By what factorial must we multiply $P_{n,r}$ to obtain $(n!)$?

   (b) Express $P_{n,r}$ as the quotient of two numbers, each of which is a factorial.

7. A monkey sits at a typewriter and types a "monkey-word" of five letters by touching 5 different keys in succession.

   (a) How many possible "monkey-words" of five letters are there?

   (b) How many different "monkey-words" of 26 letters each would be possible? (Leave your answer in product form.)

   *(c) How many days would he need to type a complete list of the 5-letter "monkey-words" if he typed a new word every second? (Assume the monkey to be an ideal typist who makes no mistakes and takes no banana-break until the job is done!)

8. A telephone dial has a finger hole for each of the ten digits.

   (a) How many telephone numbers, each with five digits but with no digit repeated, are possible?

   (b) How many telephone numbers, each with five digits, are possible?

9. Five players on a football squad can play either left-end or right-end. The five players may be in the lineup in how many different ways as left-end or right-end?

*10. Suppose we want to send messages in code. We use certain symbols, say n of them. (The symbols might be letters, or flags, or sounds, or designs, or any other type of symbol.)

Each message is composed of four different symbols, arranged in order. The number of possible messages which we may wish to send is 1600. What is the smallest number that n can be in order to meet the requirement?

7-3. Selections or Combinations

Whenever we have been using the word "permutation," we have been concerned, not only with the elements, but also with the arrangement or the ordering of the elements. At the very beginning of this chapter, we discussed committees in a club. In a committee such as we studied there, the members are not arranged in any particular manner. The choosing of a committee from a club is an illustration of a selection or combination.

Definition. A selection of a certain set of n objects taken r at a time is a set of r members from the total set of n objects with no regard to ordering the chosen members.

Here, n is a counting number and r is a whole number no greater than n. The number of selections of a set of n objects taken r at a time is often represented by the symbol $\binom{n}{r}$. In this unit you may read this symbol by saying: "the number of selections of n things taken r at a time," or, "the number of combinations of n things taken r at a time."

In terms of sets, we may say that $\binom{n}{r}$ is "the number of r-subsets in an n-set." By an n-set we mean simply a set of n elements. An r-subset is a set of r elements, each of which is one of these n elements.

The entries in the Pascal triangle are values of $\binom{n}{r}$. For example, from the fifth row of the Pascal triangle we find, reading from the left, that

$$\binom{5}{0} = 1, \quad \binom{5}{1} = 5, \quad \binom{5}{2} = 10, \quad \binom{5}{3} = 10, \text{ and so on.}$$

You will want to note that the new symbol we have introduced can be easily distinguished from a fractional symbol, because the new symbol does not have a bar between the two numbers and the parentheses are always written as part of the symbol.

Note: Other common symbols for the number of selections of n things taken r at a time are $C_{n,r}$ and ${}_nC_r$. You will want to be familiar with these symbols, although $\binom{n}{r}$ is to be preferred.

Exercises 7-3a

1. Write the special symbol for each of the following:
   - (a) The number of selections of 12 objects taken 7 at a time
   - (b) The number of permutations of 12 objects taken 7 at a time
   - (c) The number of combinations of m things taken 3 at a time
   - (d) The number of selections of $n + 2$ objects taken k at a time
2. Write in words the meaning of each of the following symbols: $\binom{6}{2}$; $P_{8,4}$; $52!$; $\binom{52}{13}$; $P_{9,7}$; $\frac{8}{4}$.
3. Use the Pascal triangle to find each of the following:
   - (a) $\binom{6}{2}$ and $\binom{6}{4}$
   - (b) $\binom{7}{4}$ and $\binom{7}{3}$
   - (c) $\binom{8}{3}$ and $\binom{8}{5}$
   - (d) $\binom{3}{1}$ and $\binom{3}{2}$
   - (e) $\binom{8}{2}$ and $\binom{8}{6}$

4. Suppose that a and b are two counting numbers and let S be the sum a + b. What important relationship between $\binom{S}{a}$ and $\binom{S}{b}$ is suggested by Problem 3? (Use some ideas in Section 7-1 to convince yourself that this relationship is true in every case.)

5. (a) Find each of the following:

   $\binom{4}{4}$, $\binom{6}{6}$, $\binom{9}{9}$, $\binom{1247}{1247}$.

   (b) What general notion do these examples illustrate?

6. (a) Find each of the following:

   $\binom{4}{0}$, $\binom{6}{0}$, $\binom{7}{0}$, $\binom{249}{0}$.

   (b) What general notion do these examples illustrate?

7. Show that if n is a counting number different from one, then $\binom{n}{2} = \frac{n(n-1)}{2}$.

Suppose that a club of seven members picks three officers. With the aid of the Pascal triangle, we learned that the number of possible selections of an executive committee is 35. This number 35 we may now call $\binom{7}{3}$. Our study of permutations tells us that the three offices may be matched with the three officers in $P_{3,3} = 3!$ ways. We may apply the Multiplication Property and see that the number of possible officer assignments is $\binom{7}{3}\cdot P_{3,3}$. The first event is the selection of a group of 3 from 7 members. The second event is the arrangement of these 3 in the offices.

On the other hand, we may apply the Multiplication Property to find that the number of choices of three members, arranged by office, from the club of seven members is $7\cdot 6\cdot 5$, namely 210. This number, 210, is $P_{7,3}$.

Both viewpoints yield the same count. Each expression represents the total number of possible officer assignments. Therefore,

$$\binom{7}{3}\cdot P_{3,3} = P_{7,3} \quad \text{or}$$

$$\binom{7}{3}\cdot 3! = P_{7,3}$$

This last equation gives us a way to calculate $\binom{7}{3}$, the number of selections of 7 different things taken 3 at a time. For we can see from the preceding equation that

$$\binom{7}{3} = \frac{P_{7,3}}{3!}$$

The same type of argument shows that for two counting numbers n and r, with $r \leq n$, we can write

$$\binom{n}{r}\cdot P_{r,r} = P_{n,r}$$

$$\binom{n}{r}\cdot(r!) = P_{n,r}$$

In words, this expression $P_{n,r} = \binom{n}{r}\cdot P_{r,r}$ simply states that

$$\left\{\begin{array}{l}\text{the number of}\\ \text{permutations of}\\ n\ \text{different}\\ \text{things arranged}\\ r\ \text{at a time}\end{array}\right\} = \left\{\begin{array}{l}\text{the number of}\\ \text{selections of}\ n\\ \text{different things}\\ \text{taken}\ r\ \text{at a}\\ \text{time}\end{array}\right\} \cdot \left\{\begin{array}{l}\text{the number of}\\ \text{permutations of}\\ r\ \text{different}\\ \text{things arranged}\\ r\ \text{at a time}\end{array}\right\}.$$

From this general equation we see that

$$\binom{n}{r} = \frac{P_{n,r}}{r!}$$

and since

$$P_{n,r} = n(n-1)\ \ldots\ (\text{to}\ r\ \text{factors})$$

we obtain the formula

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ \ldots\ (\text{to}\ r\ \text{factors})}{r!}.$$

In this fraction, the number of factors in the numerator is r, the same as the number of factors in the denominator. For example, when $n = 11$ and $r = 5$ we have

$$\binom{11}{5} = \frac{11\cdot 10\cdot 9\cdot 8\cdot 7}{5\cdot 4\cdot 3\cdot 2\cdot 1}.$$

Notice that there are r, five, factors in both numerator and denominator. The first factor in the numerator is n, eleven; the first factor in the denominator is r, five.

Exercises 7-3b

1. Ten men are qualified to run a machine that requires three operators at a time. How many different crews of three are possible?
2. A disc jockey had a set of 15 records. Each night he selects 5 records to make a program. How many nights can he do this without repeating an entire program? Disregard the order in which the individual records occur within a program. (You do not need to perform any multiplications, but may leave your answer in whatever symbols you think are convenient.)
3. Eight points are given in space, and no four of them lie in the same plane. (Remember that any three of them determine a plane.) How many different planes are determined by the eight points?
4. On a certain railway there are 12 stations. How many different kinds of tickets should be printed to provide tickets between any two stations:
   (a) In case the same ticket is good in either direction?
   (b) In case different tickets are needed for each direction?
5. A restaurant has prepared 4 kinds of meat, 3 kinds of salad, and 5 kinds of vegetables. A platter consists of a meat, a salad, and a vegetable. How many different kinds of platters are possible?
6. A girl has four skirts, six blouses, and three pairs of shoes. How many weeks can pass while she wears a different costume every day?

7. In a game of bridge, a hand consists of 13 cards from the playing deck of 52 cards. The number of possible bridge hands is 635,013,559,600. Write this number, using a special symbol you have studied in this unit.

8. A salesman has customers in eight cities away from his home. He wishes to plan a travel route which will take him to each of the eight cities in turn and afterwards back to his home. How many possible routes are there?

*9. There are eight teams in a baseball league. During the season each team played every other team five times. How many games are played in the league altogether during one season?

*10. Either one bulb or two bulbs of a string of eight Christmas tree lights wired in series are burned out. Suppose you have two good bulbs and suppose you try, first one at a time, then two at a time, to locate the burned out bulb (or bulbs). How many trials might it be necessary for you to make in order to find the bulb (or bulbs) that need replacement?

*11. A man has six bills, one each of the amounts of $1, $5, $10, $20, $50, $100. How many different sums of money may be formed by using one or more of these six bills together?

7-4. Review of Permutations and Selections

In this chapter we have studied ways of counting all possible arrangements, or permutations, of a set of elements and of counting all possible subsets of a given size. We have called one such subset a selection, or combination. It is important that you keep in mind the terms, permutations and selections, and that you understand the principles on which the counting methods are based. It is not so important at this level of your study of mathematics that you remember particular formulas.

In the exercises of this section the basic ideas associated with permutations and selections may be applied.

Exercises 7-4

1. How many "donkey" words of six letters each can be formed from the letters of the word THEORY? ("Donkey" words need not make sense.)

2. A girl bought two new skirts and three new sweaters. How many different outfits consisting of a new skirt and a new sweater can she wear?

3. Ten years after graduation a class held a reunion to which 96 persons came. If each person shook hands with everyone present, how many handshakes were there?

4. A certain make of automobile has 3 body types, 7 choices of upholstery, and 5 color schemes. In order to show all possible cars at an exhibit, how many cars are necessary?

5. Five Indians walked one behind the other in the woods. Into how many different orders could they place themselves?

6. From 14 men how many different committees of 5 can be formed?

7. A troop of Sea Scouts has 8 different flags. How many different signals can they send by flying 3 flags at a time on a pole?

8. The call letters of a certain broadcasting station begin with W. How many different call letters using only 3 letters can be used? Repetition of a letter is allowed.

9. How many automobile license plates bearing just 5 digits can be made if 0 is not permitted as the first digit?

10. A housewife wishes to arrange 5 books on her desk. She has 8 books from which to choose. The 8 books have different colored covers and are of different size. Does she have a problem involving permutations or selections? What is the number of different patterns that she can look at?

11. A housewife wishes to read 5 books in the next two weeks. There are 8 books from which to choose. Does she have a problem involving permutations or selections? How many different sets of five books may she choose to read?

12. Three points determine a plane and 2 points determine a line. From five points no four of which are in the same plane and no three of which are on the same line,

    (a) How many planes will be determined?

    (b) How many lines will be determined?

Chapter 8

PROBABILITY

8-1. Chance Events

This chapter will be concerned with chance events. For example, a weatherman makes a forecast of the future weather. His forecast, "Rain," is more accurately a probability statement, "It will probably rain." Similarly, you may predict that "The Green Shirts will win the pennant," but what you mean to say is "It is likely that the Green Shirts will win the pennant."

Probability has many practical uses. For example, federal and state governments use probability in setting up budget requirements; military experts use it in making decisions on defense tactics; scientists use it in research and study; engineers use probability in designing and manufacturing reliable machines, planes and satellites; big business companies use it in mathematical studies to help make difficult decisions; insurance companies use it in setting up life expectancy tables.

Some examples of games of chance will be used to help you understand what probability means and how it may be used. Such games give us excellent mathematical models for use in studying probability. The examples are not used with the idea that gambling is to be encouraged. Rather, the information in this chapter should help you begin to understand why "most gamblers die broke."

In Section 1 we shall study some ideas about statements involving chance events, like "The Brown Sox will win," or "Sandlot will win the race," or "If I toss a coin and allow it to fall freely, it will show heads." We will concern ourselves with a "measure of chance" that an event will happen. This measure of chance is also called the probability that the event will occur. At first we will use a mathematical model where we can count the possible outcomes. The game of tossing a coin can be used as a model. If we toss a coin and allow it to fall freely, either a head will show or a tail will show. We assume the coin is perfectly balanced and that neither side is weighted in any way. Such a perfectly balanced coin is sometimes called an "honest coin."

Consider the question, "What is a measure of chance that if we toss a coin and allow it to fall freely a head will show?"

In probability it is useful to use a number to indicate the measure of chance that an event will happen. If we toss a coin we consider two possible outcomes: (1) a head will show or (2) a tail will show. That the coin will show a head is one favorable outcome out of two possible outcomes. We say the measure of chance that the coin will show a head is $\frac{1}{2}$.

If an event is governed by chance, then it has a certain probability of happening. If we use the letter "A" to represent the event that the coin will show a head, then we can call $\frac{1}{2}$ the probability of the event A. This is the same as saying that the measure of chance that the event will occur is $\frac{1}{2}$. We can represent the probability of the event A as

$$P(A) = \frac{1}{2}.$$

If we use the letter "B" to represent the event that the coin will show a tail, we are concerned with the probability of B. We can represent this with the symbol P(B). Thus,

$$P(B) = \frac{1}{2}.$$

It is important that you understand that in the above case $P(A) = P(B)$. That is, each event is equally likely to occur. Any two statements which predict events that are equally likely have the same probability.

Suppose you have tossed an honest coin five times and it shows a head each time. What is the probability that the coin will show a tail on the next toss? Some people believe that the odds will change, that the "forces of luck" will act to force the coin to show a tail until a balance is restored between the showing of heads and tails. Not so! The probability that the coin will show heads remains $\frac{1}{2}$ for each toss. In probability we do not say that if the coin shows a head on the first toss it must show a tail on the second toss.

Suppose you use two pennies. What is the measure of chance that if two coins are tossed, one head and one tail will show? That is, what is the probability that the event of one head and one tail showing will occur? The table below shows that there are four possible outcomes:

Possible Outcomes

First Coin	Second Coin
Head	Head
Head	Tail
Tail	Head
Tail	Tail

There are two outcomes showing one head and one tail. Two outcomes out of four possible outcomes are favorable. The probability that the event will occur is $\frac{2}{4}$ or $\frac{1}{2}$. If we use the letter "E" to represent the event, we may write

$$P(E) = \frac{1}{2}.$$

What is the probability that exactly two heads will show if two coins are tossed? It does not make any difference whether the coins are tossed at the same time or one following the other. Of the four outcomes, how many ways are there for this event to occur? If we use the letter "G" to represent the event that two heads show, we may write the probability of the event G as

$$P(G) = \frac{1}{4}.$$

Note that in this example, events E and G are not equally likely. Their probabilities are different.

We may write the formula:

$$P(E) = \frac{t}{s},$$

where $P(E)$ is the probability that an event E will occur, t is the number of possible outcomes in which E occurs, s is the total number of possible outcomes. If r is the number of possible outcomes in which E does not occur, then we may say

$$P(\text{not } E) = \frac{r}{s}.$$

Since either E occurs or E does not occur, $t + r = s$, and

$$P(E) + P(\text{not } E) = \frac{t}{s} + \frac{r}{s} = \frac{t + r}{s} = \frac{s}{s} = 1.$$

On the assumption that an event E either occurs or does not occur, then

$$P(E) + P(\text{not } E) = 1.$$

If an event K is certain to happen, $P(K) = 1$.
If an event L cannot occur, $P(L) = 0$.
Thus, we conclude that for any event M

$$0 \leq P(M) \leq 1.$$

This number sentence is read "P(M) is greater than or equal to zero and less than or equal to 1."
Why is P(M) never greater than 1?

Exercises 8-1a

1. Two black marbles and one white marble are in a box. Without looking inside the box, you are to take out one marble. Find the probability of the event that when, without looking, one marble is taken out of the box, the marble will be black.
2. Using the data in Problem 1, find P for the event that if one marble is taken out of the box, it will be white.
3. Suppose you have tossed an honest coin nine times and it has shown a head each time.
   (a) Consider the above as one event. Is this event likely to occur? Explain your answer.
   (b) What is the probability that the coin will show a tail on the tenth toss?
   (c) Does the outcome of the first 9 tosses have any effect on the outcome of the tenth toss?

4. There are 25 students in a class, of whom 10 are girls and 15 are boys. The teacher has written the name of each pupil on a separate card. If a card is drawn at random, what is the probability that the name written on the card is:
   (a) the name of a boy?
   (b) your name (assuming you are in the class)?

5. Suppose a box contains 48 marbles. Eight of the marbles are black and forty of the marbles are white. Find P for the event that if a marble is picked at random (without looking in the box), it will be white.

6. Using the data for Problem 5, consider the event: "If, without looking, nine marbles are taken out of the box, all of the marbles will be black."
   (a) Is the outcome in this case possible?
   (b) What measure of chance can we assign to such an outcome?

7. From a large amount of evidence, we know that boys and girls are born in about equal frequencies. On the average, half of all babies born are boys, and half are girls. In a given birth, then, the probability of a baby being a boy is $\frac{1}{2}$. Likewise the probability of its being a girl is $\frac{1}{2}$, since no other outcomes are possible. (We exclude for the moment the possibilities of twins, triplets and other multiple births.) Then, $P(\text{boy}) = \frac{1}{2}$ and $P(\text{girl}) = \frac{1}{2}$. Let us assume that these measures of chance hold for any particular family as well as in general.
   (a) Mr. and Mrs. Jones already have one boy when the second baby arrives. What is the probability of its being a boy? A girl? (It is important to remember that each birth is an independent event, and not influenced by previous births. We agree that the fact that the Joneses already had a boy does not affect the probability of the second baby being a boy or a girl.)

(b) Mr. and Mrs. Richards have eight children, all girls, when the ninth baby arrives. What is the probability of its not being a girl?

8. In a newspaper you read: "He has a 50-50 chance of winning the election."
   (a) What is the probability that he will win?
   (b) Suppose a measure of chance is less than $\frac{1}{2}$. What does this mean in terms of the outcome of an event? Is the outcome very likely or not very likely to occur?

9. If a whole number from 1 to 30 (including 1 and 30) is selected, what is the probability that the number will be a prime number? Assume that the selection is made so that one number is just as likely to be chosen as any other.

10. Three hats are in a dark closet. Two belong to Mr. Smith and the other to his friend. Being a polite person, when his friend is ready to leave with him, Mr. Smith reaches in the closet and draws any two hats. What is the probability that he will pick two wanted hats, his friend's hat and one of his own hats?

11. Suppose you have five cards, the ten, jack, queen, king, and the ace of hearts.

   (a) What is the chance that the first card you draw is the ace?

   (b) Assume that you draw the jack on the first draw, and put it aside. What is the chance that the second card you draw is the ace?

   (c) Are your answers for (a) and (b) the same? Why?

   (d) After drawing the jack, and putting it aside, assume that the second card you draw is the ten. Put that aside also. What is the chance that the third card you draw is the ace?

   (e) What is true of the measure of chance of the drawings in (a), (b), and (c) as the number of cards decreases?

In some of the problems you determined the measure of chance, which we call probability, by listing all possible outcomes. This is easy when there are only one or two coins, but as the number of coins increases, it is difficult to remember all the possibilities. Let us see if we can discover an easy, accurate way to make these listings.

The table for two coins shows this pattern:

Possible Outcomes

First Coin	Second Coin
H	H
H	T
T	H
T	T

("H" represents heads and "T" represents tails.)

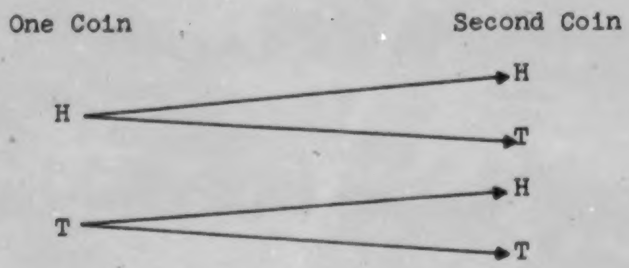
Note that the first column is grouped by twos; H, H, T, T. The second column is grouped alternately; H, T, H, T. Compare the pattern in the table for two coins with the pattern in the table for three coins shown below:

Possible Outcomes

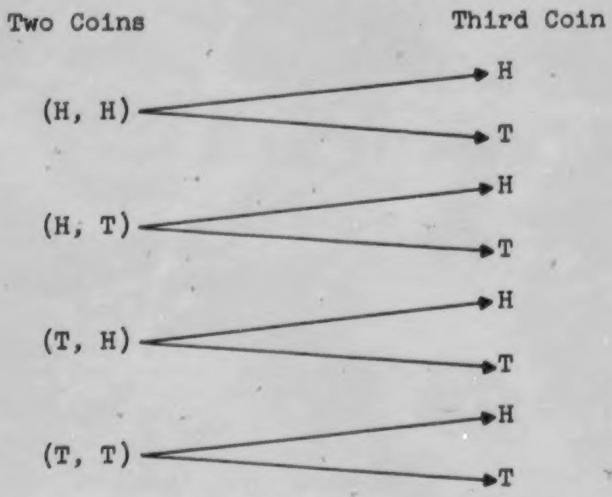
First Coin	Second Coin	Third Coin
H	H	H
H	H	T
H	T	H
H	T	T
T	H	H
T	H	T
T	T	H
T	T	T

Note how each column is grouped: the first by fours; the second by twos; the third alternately, H and T. This is one systematic way in which the number of possible outcomes might be listed in order to count the possibilities.

How many possibilities are there for one coin? You know that there are only two, H or T. What did you find for the possibilities when two coins are tossed? There were twice as many possibilities because for each possibility for one coin there were two possibilities for the second coin. This is pictured in the following diagram.



If a third coin is added should the number of possibilities be doubled again? The following diagram may help you decide.



For each possible arrangement for two coins, there are two possibilities for the third coin. Thus, the number of possibilities for three coins is equal to 2 × (the number of possibilities for 2 coins). In general you should now understand that each time one more coin is used the number of possibilities is then doubled. How many possibilities would there be if four coins were used? Recall that there were eight possibilities with 3 coins.

In summary:

Number of coins	Total number of possibilities
1	2
2	$2\cdot 2 = 2^2 = 4$
3	$2\cdot 2\cdot 2 = 2^3 = 8$
4	$2\cdot 2\cdot 2\cdot 2 = 2^4 = 16$
⋮	⋮
10	$2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2 = 2^{10} = 1024$
⋮	⋮
n	$\overbrace{2\cdot 2\cdot 2\cdot \ldots \cdot 2}^{n} = 2^n$

(Notice that each entry in the right column is twice that above it.)

We can express this result as a formula:

$$T = 2^n$$

T is the total number of possibilities,
2 is the number of possibilities for one coin,
n is the number of coins.

*Some of you may want to try to justify the general formula of this type:

$$T = s^n$$

T is the total number of possibilities,
s is the number of possibilities for one object,
n is the total number of objects used.

If the possibilities of an object are A, B, and C, what would be the number of possibilities for two of these objects?

$$T = s^n$$
$$T = 3^2$$
$$T = 9$$

There are nine possibilities for two objects, each having three possibilities.

In making tables, as in the cases of tossing two or three coins, all possible outcomes of events were listed. The probability is based on the outcomes listed in the table. In the tables discussed we assume that each separate possibility, or outcome, has the same chance of occurring. We say that each outcome is "equally likely" to occur.

In this section we have been concerned with some simple events governed by chance. We assigned measures of chance, which we called probabilities, for the outcomes of these events. The numbers we used to represent "P" were numbers like one-half, two-thirds, one-fourth, and so on. If we actually toss an honest penny once, we cannot predict whether it will show a head or a tail. But if we toss an honest penny a million times, then it is almost certain that the number of heads will be between 490,000 and 510,000. The ratio of heads shown to the number of tails shown almost certainly will be between $\frac{49}{100}$ and $\frac{51}{100}$. We cannot in this chapter study all the mathematics that is required as a basis for such conclusions.

It should be kept in mind that probability is not the tossing of coins or drawing of cards. Probability is a part of mathematics which has been found exceedingly useful in describing chance aspects of games, selections, science, business, and activities of government which are not completely predictable. In this chapter we will study some of the more elementary ideas of this mathematical theory.

Exercises 8-1b

1. If three honest coins are tossed, what is the probability that three heads will show? Refer to the table in the previous section showing 8 possibilities for 3 coins.

2. If three honest coins are tossed, what is the probability that two heads and one tail will show?

3. Without listing them, determine the number of possible outcomes in tossing five coins.

4. There are 35 bricks, of which five are gold. What is the chance that if you pick a brick at random you will pick a gold one? ("At random" in this case means "without looking and without lifting.")

5. (a) If one penny is tossed, what is the chance that a head will show?

   (b) How many heads might you reasonably expect to get if the penny is tossed 50 times?

6. A bowl contains five white marbles, three black marbles and two red marbles.
   (a) What is the chance that you will pick a white marble in one draw?

   (b) Assuming you pick a white marble the first time and do not replace it, what is the chance that you will pick a black marble the second time?

   (c) Assuming you pick a white marble the first time and a black marble the second time and do not replace them, what is the chance that you will pick a red marble the third time?

7. The letters A, B, C, D, E, and F are printed on the faces of a cube (one on each face).
   (a) If one cube is rolled, how many possible outcomes are there? We will consider the side facing up as the outcome in this case.

(b) If two cubes are rolled at the same time, how many outcomes are there?

(c) What is the chance that B will show if one cube is rolled?

(d) What is the chance that two E's will show if two cubes are rolled at the same time?

8. A regular tetrahedron is a solid having four faces. The letters A, B, C, and D are printed on the faces.

(a) If a regular tetrahedron is rolled (or tossed in the air and allowed to fall freely), how many possible ways are there for it to stop (or fall)? Note that in this case we will consider the side on which the object rests as showing the outcome. That is, the face that is the base may be marked A, B, C, or D.

(b) Find the measure of chance for the following statement: "If the tetrahedron is rolled it will stop on side A."

(c) How many possible outcomes are there if two such tetrahedrons are rolled?

(d) How many possible outcomes are there if three such tetrahedrons are rolled?

9. Notice the pattern that is involved in a count of the number of outcomes in tossing coins. H is a head; T is a tail; (H,T) is a head and a tail in either order.

1 coin 1(H) 1(T)
2 coins 1(H,H) 2(H,T) 1(T,T)
3 coins 1(H,H,H) 3(H,T,T) 3(T,H,H) 1(T,T,T)

Add a fourth and fifth line in this table.
Why does this remind you of Pascal's triangle?

10. Use Problem 9 to find the probability of getting two heads and two tails if four coins are tossed.

11. Give the probabilities of each of the six possible outcomes when five coins are tossed. Is their sum one?

12. If five coins are tossed, what combinations of heads and tails are most likely to occur? Why? Hint: See Problem 9.

*13. When six coins are tossed, what is the chance that one and only one will show heads?

8-2. Empirical Probability

Among the most important applications of probability are those in situations where we cannot list all possible outcomes. For example, the table shows a small number of weather forecasts, only those from April 1 to April 10. The actual weather on these dates is also given.

Date	Forecasts	Actual weather	"Yes" indicates the forecasted event did occur, "No" that it did not.
1.	Rain	Rain	Yes
2.	Light showers	Sunny	No
3.	Cloudy	Cloudy	Yes
4.	Clear	Clear	Yes
5.	Scattered showers	Warm and sunny	No
6.	Scattered showers	Scattered showers	Yes
7.	Windy and cloudy	Overcast and windy	Yes
8.	Thunder-showers	Thundershowers	Yes
9.	Clear	Cloudy and rain	No
10.	Clear	Clear	Yes

Observe that forecasts 1, 3, 4, 6, 7, 8 and 10 were correct. We have observed ten outcomes. The event of a correct forecast has occurred seven times. Based on this information we might say the probability that future forecasts will be true is $\frac{7}{10}$. This number is the best estimate that we can make from the given information. In this case, since we have observed such a small number of outcomes, it would not be correct to say that our estimate of P is dependable. A great many more cases should be used if we expect to make a good estimate of the probability that a weather forecast will be accurate. You will understand, of course, that there are a great many other factors which affect the accuracy of a weather forecast. The example here merely indicates something about how successful a particular weather office has been in making weather forecasts--in this case, only for a small number of days.

On September 15, a major league player A had a batting average for the season of 0.387 and player B had a season average of 0.208. Based on these averages, we would expect that there is a better chance that A would make a hit the next time he is at bat, than that B would make a hit. We might even say that a measure of the chance (probability) that A would make a hit is 0.387 and that a measure of the chance that B would make a hit is 0.208.

A physicist cannot trace the motion of a single molecule of oxygen in a room, but he can estimate the probability that an oxygen molecule will hit one of the walls in a room in the next second. To draw such a conclusion requires an understanding of much more mathematics than we can study in this chapter.

In modern industry probability now plays an important role in many activities. Quality control and the reliability of a manufactured article have become extremely important considerations in which probability is used. Questions of reliability can become very complex. A basic idea related to reliability, however, can be illustrated as follows. Many thousands of articles of a certain type are manufactured. The company selects 100 of these articles at random and subjects them to very careful tests.

In these tests it is found that 98 of the articles meet all measurement requirements and perform satisfactorily. This suggests that $\frac{98}{100}$ is a measure of the reliability of the article. One might expect that about 98% of all of the articles manufactured by this process will be satisfactory. The probability or a measure of chance that an article made by this process will be satisfactory might be said to be 0.98.

All of these examples of empirical probability are different from examples and problems in Section 8-1 in one very important respect. In Section 8-1 we could list and count all possibilities except in Problem 7 of 8-1a. In this section, we cannot or it is not practical to try to do so. We draw conclusions in the first section from counting what might be called the total collection of all possibilities. In this section we draw conclusions about what may happen in the future from information we have about a sample. The selection of a sample and the size of a sample that should be used are problems of statistics. In this kind of application the selection of a sample is very important. Mathematical theories of sampling are too advanced for our consideration here.

In the problems of this section you are asked to find measures of chance or probabilities from observed data. In each case the observed data may be said to be a sample of a total population, or a sample of possible outcomes.

Exercises 8-2

1. A teacher has taught eighth grade mathematics to 1600 students during the past 10 years. In this period he has given A's to 152 students.
   (a) Based on these data what is a measure of chance that a student selected at random will receive an A in this teacher's class?
   (b) If this teacher will teach 2000 students in grade 8 mathematics during the next twelve years, how many A's might you expect the teacher to give?

2. The batting average of a baseball player is 0.333. Using this information as a measure of chance, what is the probability that this man will make a hit the next time he is at bat?

3. The record of a weather station shows that in the past 120 days its weather prediction has been correct 89 times. Use this information to state the probability that its prediction for tomorrow will be correct.

4. A manufacturer of pencil sharpeners tests carefully a sample of 500 sharpeners to see if a pencil of a certain type can be sharpened without breaking the point. In the test 489 of the tested sharpeners worked satisfactorily. There were 20,000 sharpeners in this job lot.
   - (a) What is the probability that a sharpener selected at random from the remaining 19,500 sharpeners will perform satisfactorily?
   - (b) If your school buys 40 of these sharpeners, are all of the sharpeners likely to be satisfactory?

5. Car insurance rates are usually higher for male drivers under the age of 25. Explain how by collecting data on accidents, insurance companies have found it advisable to charge a higher rate for young male drivers.

6. Life insurance and life annuity rates are based on tables of mortality. A table of mortality includes statistical data presumably giving data on 100,000 people who were alive at age 10. The following are ten lines from the Actuaries Table of Mortality.

Age	Number living	Number dying during next year	Age	Number living	Number dying during next year
10	100,000	676	40	78,653	815
12	98,650	672	50	69,517	1,108
13	97,978	671	60	55,973	1,698
14	97,307	671	70	35,837	2,327
21	92,588	683	99	1	1

According to the table, 676 of the 100,000 will not be alive at age 11. 97,978 of the original 100,000 are alive at age 13, but 671 of these persons, according to the table, die within one year.

(a) How many are alive at the age of 50?

(b) How many are alive at the age of 100?

(c) Would improved knowledge of health and medicine tend to make a table of mortality out-of-date? Why?

In Problems 7 through 10, use the Actuaries Table of Mortality given in Problem 6. Find answers correct to the nearest 0.01.

7. (a) What is the probability that a person who is 13 years of age will be alive at the age of 21?

$$\text{Hint:}\quad P = \frac{92,588}{97,978}.$$

(b) What is the probability that a person who is 13 years of age will be alive at the age of 70?

8. (a) What year was 90 years ago?

(b) Do you think a table of mortality would be very useful if it actually were constructed by selecting 100,000 people at age 10 and keeping data on them for 90 years?

*(c) By what method other than that suggested in (b) might such a table be constructed?

9. (a) What is the probability that a boy who is 10 years of age will live to the age of 99?

(b) What is the probability that a man who is 40 years of age will live to the age of 50?

10. One kind of life insurance policy guarantees to pay $1000 to a man's wife if he dies within a certain ten-year period. Would such a policy be more expensive for a man aged 40, 50, or 60? Why?

11. Consider the following events:

Event A. It rains on Friday, the 13th.

Event B. The sun shines all day on Friday, the 13th.

The following table shows the weather on twenty Friday, the 13ths. Using the information listed in the table, find P for the events A and B. Based on the information in the table, which is more likely to occur over a great number of Friday, the 13ths, A or B ? Note that it is possible that neither event occurs.

Weather on Twenty Friday, the 13ths

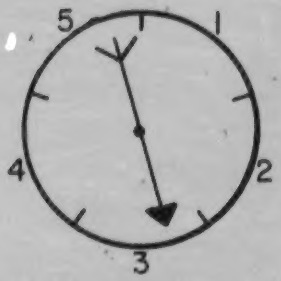
1. Heavy rain
2. Light rain
3. Sunny
4. Sunny
5. Sunny
6. Scattered showers
7. Showers
8. Sunny
9. Sunny
10. Sunny
11. Cloudy, no rain
12. Partly cloudy
13. Cloudy with some showers
14. Showers
15. Sunny
16. Sunny
17. Hot and sunny
18. Sunny
19. Cloudy and some showers
20. Sunny

8-3. Probability of A or B

In mathematics we are always looking for general principles which describe a certain situation. In this section and the next we will identify two of the most important general principles of probability.

Consider the following problem.

A dial and a pointer like the one illustrated will be used for the problem. The pointer spins and we can tell whether it stops at 1, 2, 3, 4, or 5. What is the probability that the pointer will stop at an even number?



In the figure the pointer is at 3. We shall say the pointer is at 3 if it stops between the marks on either side of 3. In order to have each spin of the pointer count we shall say the pointer is at 3 if it stops on the mark separating 3 and 4. Similarly, if it stops on the mark separating 5 and 1 we shall say it is at 5.

There are five possible outcomes. The pointer can stop at 1, 2, 3, 4, or 5. The event, the pointer stops at an even number, occurs if the pointer stops at 2 or 4, that is--the event occurs in two out of five possible outcomes. Thus the probability of the hand stopping at an even number is $\frac{2}{5}$.

The event that the pointer stops at an even number is actually a combination of two other events. Let A be the event of the pointer stopping at 2, and B be the event of the pointer stopping at 4. If we use the symbol "A or B" to stand for the event either A or B occurs, then "A or B" is the event of the pointer stopping at an even number. We have found that

$$P(A \text{ or } B) = \frac{2}{5}.$$

Could we find this probability by considering events A and B separately? We know that

$$P(A) = \frac{1}{5}. \quad \text{Why?}$$

$$\text{and} \quad P(B) = \frac{1}{5}. \quad \text{Why?}$$

If we add $\frac{1}{5}$ and $\frac{1}{5}$, the result is $\frac{2}{5}$. How can we obtain P(A or B) from P(A) and P(B)?

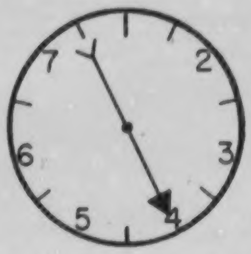
$$\frac{2}{5} = \frac{1}{5} + \frac{1}{5}$$

In this example P(A or B) = P(A) + P(B).

Our intuition certainly tells us that the probability that the pointers will stop at 2 or at 4 is greater than the probability that the pointer will stop at 2 and greater than the probability that the pointer will stop at 4. Many times (as in the above case) we can add probabilities of individual events to find the probability of another event. Notice that in the case above, the pointer could not stop at 2 and 4 at the same time (as a result of one spin). For one spin it had to stop at one or the other. Events A and B could not both occur at once. This is one of the conditions that must be met before we can add probabilities. Two events which cannot occur at once are called mutually exclusive events.

Let us consider another example.

The seven numbers are equally spaced.



The pointer spins freely. What is the probability that it will stop at an even number?

There are 7 possible outcomes. Three of the 7 are favorable outcomes. We shall call these favorable outcomes events A, B, and C: A, the pointer stops at 2; B, it stops at 4; C, it stops at 6. The event whose probability we seek is A or B or C. Events A, B, and C are mutually exclusive, since the hand can stop at only one of the numbers as a result of one spin.

Therefore,

$$P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C).$$

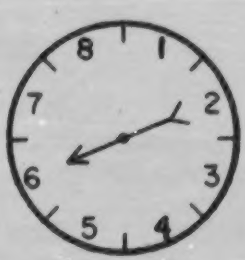
$$P(A) = \frac{1}{7} \text{ Why? Also } P(B) = \frac{1}{7} \text{ and } P(C) = \frac{1}{7}.$$

$$P(A \text{ or } B \text{ or } C) = \frac{1}{7} + \frac{1}{7} + \frac{1}{7} = \frac{3}{7}.$$

Note that we draw the conclusions, P(A or B) = P(A) + P(B) and P(A or B or C) = P(A) + P(B) + P(C), each from a single example. You could check these conclusions by solving some of the problems in Exercises 8-3 by both of the methods considered in the example at the beginning of this section.

Exercises 8-3

1. On the dial, the numbers are equally spaced around the dial.



What is the probability that the spinning pointer will stop at an odd number?

2. (a) What is the probability of obtaining a 6 or a 1 on one roll of a cube with faces numbered 1 through 6?

   (b) What is the probability of not getting a 6 or a 1 on one roll of a cube with faces numbered 1 through 6?

3. (a) What is the sum of the probabilities in Problem 2(a) and (b)? Can you interpret this as the probability of an event that is certain to happen?

   (b) Could you use the probability which you have obtained in Problem 2(a) to solve Problem 2(b)?

4. In a bag there are eight white marbles and two red marbles. If a marble is selected at random, what is the probability

of not selecting a red marble in one draw?

5. Let A be any event. Let B be the event "A does not occur." Write an equation which relates P(A) and P(B).

6. In a bag there are four red, three white, and two blue marbles. If a marble is selected at random,
   (a) What is the probability of getting a red marble?
   (b) What is the probability of getting a white marble?
   (c) What is the probability of getting a red or a white marble?

7. In a neighborhood pet show there are ten dogs, eight cats, three canaries, and six rabbits. A special prize will be given to an owner of a pet by drawing one name of an owner from the set of entry blanks.
   (a) What is the probability that the owner of a dog or a cat will get this prize?
   (b) What is the probability that the owner of a four-legged pet will not get this prize?

8. Mutually exclusive events are events which cannot happen at the same time. If one event happens, the other cannot. With this in mind, which of the following events are mutually exclusive:
   (a) The event of throwing a head or a tail on a single toss of a coin.
   (b) The event of your solving exactly six problems on a test or your solving eight problems on the test.
   (c) The event of rolling an odd number or of rolling a 3 on a die.
   (d) The event of rolling a 6 or a 3 on a cube with faces numbered 1 to 6.
   (e) The event of driving the car or going to the store.

(f) The event of going upstairs or going downstairs.

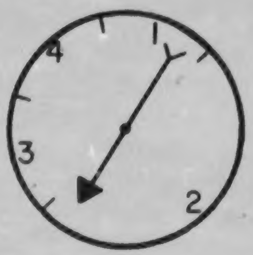
(g) The event of drawing an ace or a jack from a deck of cards on a single draw.

(h) The event of running or sitting.

(i) The event of talking to your teacher or of talking to your mother, if you talk only to one person.

(j) The event of stalling the car or of starting the car.

9. The dial is divided so that one-half of the circle is allowed for 2; 1, 3, and 4 are equally spaced.



What is the probability that the spinning pointer will stop at an even number?

10. In a bag there are four red cards and five black cards.

(a) How many different pairs of cards are there in the bag? Hint: $\binom{9}{2}$.

(b) How many different pairs of red cards are there in the bag?

(c) How many different pairs of black cards are there in the bag?

(d) How many different pairs are there in the bag consisting of a red card and a black card?

(e) How does the sum of the numbers of pairs in (b), (c) and (d), compare with the number of pairs in (a)?

11. Find the probability of the following drawings of cards from the bag in Problem 10.

(a) A pair in which both cards are the same color.

(b) A pair consisting of a red and a black card.

12. Call C the event of getting a head on at least one of the coins when two coins are tossed. Call A the event that a head will show on the first coin and B the event that a head will show on the second coin. Then

$$P(A) = \frac{1}{2} \text{ and } P(B) = \frac{1}{2}.$$

(a) Why is $P(C) = \frac{3}{4}$?

(b) Can event A and event B happen at the same time?

(c) In this problem why is it true that

$$P(A) + P(B) \neq P(C)?$$

8-4. Probability of A and B

In Section 8-3 we found the probability that either event A or event B occurs. If it is impossible for A and B to happen at the same time (A and B are mutually exclusive events) the probability of A or B is the sum of the probability of A and the probability of B. In symbols, we write

$$P(A \text{ or } B) = P(A) + P(B).$$

We now want to find the probability that both of two events will occur. What is the probability that if two coins are tossed both will show heads? The possible outcomes are: (H,H), (H,T), (T,H), and (T,T). Hence, the probability that both coins will be heads is $\frac{1}{4}$. If we call event A the event that one coin shows heads, and B the event that the other coin shows heads, then

$$P(A) = \frac{1}{2} \text{ and } P(B) = \frac{1}{2}.$$

$$P(A \text{ and } B) = \frac{1}{4}. \text{ Note that } \frac{1}{4} = (\frac{1}{2}) \cdot (\frac{1}{2}).$$

A and B is the event that both coins show heads. For this example, we see that $P(A \text{ and } B) = P(A) \cdot P(B)$. It should be observed that events A and B are independent events. Whether one coin shows heads or tails has no effect whatsoever on the other coin.

Consider the tossing of a coin and the spinning of a pointer on a dial with 1, 2, 3 and 4, equally spaced. If the coin is tossed, no matter what side of the coin appears up, this outcome has no effect upon the outcome of the spinning pointer. This is another example of independent events. If we let A be the event that a head will show when the coin is tossed and B be the event that the pointer stops at 4, then A and B are independent events.

If we wish to find the probability of a head appearing and the pointer stopping at 4 we are looking for the probability that two events will occur. If we let "A and B" stand for the event "both A and B occur" then we are looking for P(A and B).

By listing all possibilities we obtain the following:

H,1	H,4	T,3
H,2	T,1	T,4
H,3	T,2	

H,1 means that coin will show heads and the pointer will stop at one.

The desired event is H,4 which is one of eight possible outcomes. Thus

$$P(A \text{ and } B) = \frac{1}{8}\ .$$

We can also solve the problem by finding P(A) and P(B).

$$P(A) = \frac{1}{2}, \quad P(B) = \frac{1}{4}.$$

Notice that $\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$, which is the probability that we found for event (A and B).

Another way to think of this as a product is to notice that out of the favorable outcomes for A only one of the possible outcomes for B (or $\frac{1}{4}$ of the possible outcomes for B) is favorable. Hence, the probability of A and B is $\frac{1}{4}$ of P(A), and thus $P(A \text{ and } B) = \frac{1}{4} \cdot P(A) = (\frac{1}{4}) \cdot (\frac{1}{2}) = \frac{1}{8}$.

Let us think about one more example:

You are taking a test of multiple-choice questions where there are 5 choices of answers for each question. You have answered all the questions except Questions 7 and 9 which are troublesome. By elimination, you know that the correct answer for 7 is one of 2 selections, and the correct answer for 9 is one of 3 selections. You decide to guess. Find the probability of getting both 7 and 9 correct, assuming your guess on Question 7 does not affect your guess on Question 9.

Let A be the event that you choose the correct answer for Question 7, and B be the event that you choose the correct answer for Question 9. Events A and B are independent. Why? We want to know P(A and B). By the property observed in the two earlier examples,

$$P(A \text{ and } B) = P(A) \cdot P(B).$$

We also know that $P(A) = \frac{1}{2}$. Why? What does it mean to "guess"? Also, $P(B) = \frac{1}{3}$.

$$P(A \text{ and } B) = (\tfrac{1}{2}) \cdot (\tfrac{1}{3}) = \tfrac{1}{6}.$$

The probability of getting both Questions 7 and 9 correct by guessing is $\frac{1}{6}$.

Our intuition tells us that the probability that two events will happen, P(A and B), is less than the probability that one or the other will happen, P(A or B). Our intuition also tells us that P(A and B) is less than both P(A) and P(B). In this statement it is assumed that neither of the events can have a probability of 1.

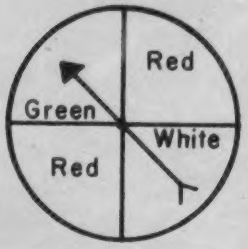
Exercises 8-4

1. You toss a coin twice in succession. Let A be the event that a tail shows on the first toss of the coin. Let B be the event that a head shows on the second toss.
   (a) Are events A and B independent? Explain.
   (b) Find the probability that the coin will show heads on both tosses.

2. (a) If A, B, and C are independent events, then P(A and B and C) = P(A) × P(B) × P(C). State a similar property which holds for four independent events.
   (b) Find the probability of a head showing on each of nine successive tosses of a coin.

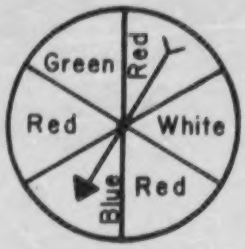
3. Your basketball team is to play team A and team B on two successive dates. It is estimated that the probability of winning over A is $\frac{1}{2}$ and over B is $\frac{2}{5}$.
   (a) What is the probability of your team winning both games?
   (b) If your team won the first game, what is the probability of winning the second?

4.



The four sections are equal.

Dial A



The six sections are equal.

Dial B

Both pointers are made to spin. Assume both are honest.

(a) What is the probability that both will stop on red?

(b) What is the probability that both will stop on green?

(c) What is the probability that A stops on white and B stops on blue?

5. If you have a bag of five black marbles and four white marbles, what is the chance of drawing two white marbles from the bag if one is drawn and then replaced before the second drawing?

6. In Problem 5, what is the chance of drawing two white marbles if the first one is not replaced before the second drawing?

7. (a) Are the events in Problem 5 independent events?

   (b) Are the events in Problem 6 independent events?

8. Assuming that the probability of the Greens having a boy is $\frac{1}{2}$, and of having a girl, is $\frac{1}{2}$,

   (a) What is the probability of the Greens having a boy and a girl as their first two children?

   (b) What is the probability of the Greens having first a boy, then a girl?

   (c) What is the probability of their having first a girl, then a boy?

   (d) If the Greens have a third child, what is the probability that it will not be a girl?

9. Which of the following pairs of events are independent?

   (a) Picking a black marble both times in two draws from a bag containing black and white marbles if you do not replace the first marble drawn.

   (b) Picking a black marble both times in two draws from a bag containing black and white marbles if you replace the first marble drawn.

   (c) Going to school and becoming a lawyer.

   (d) Throwing a 3 on a cube with numbered faces and getting a head when a coin is tossed.

   (e) The event of a day being sunny and the event of the next day being partly cloudy.

10. A certain problem is to be solved. The chance that one man will solve the problem is $\frac{2}{3}$. The chance that another man will solve the problem is $\frac{5}{12}$.

    (a) What is the chance that the problem will not be solved when both men are independently working on it?

    (b) What is the chance that it will be solved?

11. If a committee of 3 is to be chosen from a class of 20 pupils and each pupil is as likely to be chosen from a class as any other pupil, what is the chance that you and your two best friends will be chosen?

12. When six coins are tossed, what is the chance that at least one head will be obtained?

13. Almost a hundred years ago a monk named Mendel did many experiments in breeding plants, especially garden peas. The results of these experiments were so important that our modern knowledge of heredity is based on his findings.

    We now know that inherited traits are controlled by genes, and that these are located on the chromosomes. Just as a person has two chromosomes of a particular kind, such as A, he also has two genes for a particular trait. These genes need not be exactly the same. They can affect two different appearances of the same trait: brown eyes and blue eyes, or curly hair and straight hair, for example.

    It is important to know that a parent will pass along to a child only one gene of the two he has of a particular kind. Each child will receive one of the two possible genes for a trait from his mother, and one of the two possible genes for the same trait from his father. The probability of getting either one of the two genes from a parent is $\frac{1}{2}$.

    What these genes turn out to be, by chance, in the child will affect the trait. In carnations, for example, red flowers are produced when a plant has two R genes (RR), and white flowers result when a plant has two r genes (rr). But, if a plant has one R gene, and one r gene, then the flowers are pink.

(a) What is the probability of red-flowered plants producing R genes?

(b) What is the probability of white-flowered plants producing R genes?

(c) What is the probability of getting red flowers when pink-flowered plants are crossed with pink-flowered plants?

(d) What is the probability of getting red flowers when red-flowered plants are crossed with pink-flowered plants?

*14. There are ten sticks. One is an inch long, one is 2 inches long and so on up to ten inches long. A person picks up three of these sticks without looking. What is the probability that he can form a triangle with them? Remember the sum of the lengths of two sides of a triangle is greater than the length of the third side.

*15. Ten slips of paper numbered 1 to 10 are put in a hat and thoroughly mixed. Two slips of paper are drawn by a blindfolded person. What is the probability

(a) That the numbers on both slips are even?

(b) That the sum of the two numbers is even?

(c) That the sum of the two numbers is divisible by 3?

(d) That the sum of the two numbers is less than 20?

(e) That the sum of the two numbers is more than 20?

16. BRAINBUSTER.

    (a) A penny, a nickel, a dime, and a quarter are thrown and exactly two come up heads. What is the probability that one of those coming up a head is the dime?

    (b) If the same four coins are thrown and exactly three come up heads, what is the probability that one of the three is the dime?

17. BRAINBUSTER. Five different coins are thrown, a half dollar in addition to those in Problem 16. What is the probability of each of the following?

    (a) If exactly three come up heads, one is a dime and one is a quarter.

    (b) If exactly two come up heads, one is a dime.

    (c) That exactly two come up heads and one of these is the dime.

    (d) That exactly three come up heads and two of these are a dime and a quarter.

8-5. Summary

In this chapter you have studied some elementary ideas of probability, solved problems involving applications of these ideas, and made use of probability notation such as $P(A)$. You have observed that, by definition,

$$0 \leq P(A) \leq 1.$$

$P(A)$ is the probability, or measure of chance, that event A will happen.

Two types of situations in which probability can be applied have been considered. On one, you can list and count all possible outcomes. In the other you cannot, or do not because of the tedious work involved, count all possible outcomes. In this situation you choose a sample of data and then base the count on the sample rather than on the total.

You have also solved problems, illustrating

(1) P(A or B) = P(A) + P(B) where P(A or B) is the probability that event A or event B will happen, provided that A and B cannot both happen (A and B are mutually exclusive).

(2) P(A and B) = P(A) · P(B) where P(A and B) is the probability that both event A and event B will happen, provided that the success or failure of A has no effect whatsoever on B (A and B are independent).

Exercises 8-5

1. The numerals for the numbers 1 through 16 are placed on 16 disks. If one of the disks is selected without looking, what is the probability that the number named is:

   (a) divisible by 4?

   (b) divisible by 3?

   (c) a prime number?

   (d) a two-digit number?

   (e) divisible by 4 and 3?

2. A pencil box contains 5 hard pencils and 12 soft pencils. If you pick out one pencil what is the probability that it will be:

   (a) soft?

   (b) hard?

   (c) hard or soft?

3. At a signal, each of five boys tosses a penny into the air. What is the probability that all the coins will come down heads?

4. A card is drawn from an ordinary deck (52 cards).
   (a) What is the probability that the card is a diamond?
   (b) What is the probability that the card is an ace?
   (c) What is the probability that the card is a diamond and an ace?
   (d) What is the probability that the card is a diamond or a spade?

5. A father brought home a set of alphabet blocks for his two-year-old son. Each block had the same letter on each of its faces. The father selected the blocks necessary to spell the child's name, "A L B E R T," and gave them to the boy, who did not know one letter from the other. After playing with the six blocks for a while the child arranged them in a line. What is the probability that the arrangement spelled his name?

6. A bag contains 4 times as many red marbles as black marbles (identical except for color). If one marble is drawn what is the probability that it is red?

7. If there are two black marbles and one white marble in a box, we can say that there are three possible pairs of marbles in the box. Find the probability that when, without looking, a pair of marbles is taken out of the box,
   (a) both marbles will be black?
   (b) one marble will be black and one marble will be white?

8. Find the sum of the probabilities in Problem 7 (a) and (b). Explain the meaning of the sum.

9. You are to be placed in a line with two girls (or boys) one of whom is your favorite. If the line contains exactly three persons including yourself, what is the probability that you will stand next to your favorite? In such a problem we assume that you are not placed according to any plan (including your

own). If you crowd in next to your favorite, chance would not play a role.

10. In Disco Junior High School students have been divided into sections alphabetically. In Section D the numbers of students counted by the first initials of their last names are as follows:
K - 5, L - 4, M - 8, N - 4, O - 2, P - 5, Q - 1.
    (a) Find the probability that a student selected at random will have a family name beginning with K or L.
    (b) Find the probability that a student selected at random will have a family name beginning with O, P, or Q.
    (c) What is the probability that a student selected at random will not have a last initial of M or N?

11. Based on the season's records on September 1, the batting average of A is 0.313, of B is 0.260, and of C is 0.300. If A, B, and C bat in order, what is the probability that all three men will get a hit? (Round your answer to the nearest thousandth.)

12. Based on data available, biologists consider the probability of the birth of a boy $\frac{1}{2}$, and of a girl, $\frac{1}{2}$. In the birth of three children,
    (a) What is the probability that all will be boys?
    *(b) What is the probability that at least two will be boys?

13. Suppose on a regular dodecahedron, a solid having twelve plane faces, 5 faces are colored white and 7 faces colored black. If you toss it, what is the chance that it will stop with a white side down?

14. If there are 225 white marbles and 500 black marbles in a box, what is the chance of picking a black marble on the first draw?

15. On Monday the Panthers play the Bears and on Tuesday the Panthers play the Nationals. Based on the season's record it is said that the probability that the Panthers will win the Bear's game is 0.4, and the probability that the Panthers will win the National's game is 0.6. Assume that the results of the first game have no effect on the outcome of the second game.

   (a) What is the probability that the Panthers will win both games?

   (b) What is the probability that the Panthers will lose both games?

   (c) What is the probability that the Panthers will win the Bear's game and lose the National's game?

   (d) What is the other possible outcome of playing both games? What is the probability of this event?

*16. A cube with faces numbered 1 to 6 and a coin are tossed at the same time.

   (a) What is the probability that both a head and a 6 will show?

   (b) What is the probability that a head or a 5 will show?

   (c) What is the probability that a head or a number divisible by 2 will show?

*17. Suppose you have six letters to be delivered in different parts of town. Two boys offer to deliver them. In how many different ways can you distribute the letters to the boys? Include the possibilities of one boy having 0, 1, and 2 letters to distribute as well as the possibility that each boy will have 3 letters to distribute.

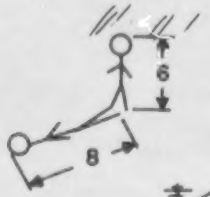
Chapter 9

SIMILAR TRIANGLES AND VARIATION

9-1. Indirect Measurement and Ratios

You read in Chapter 3 that the sun is 93,000,000 miles away from the earth, and that the distance from the earth to the nearest star (other than the sun) is 4 light-years. You probably know that the diameter of the earth is about 8,000 miles. Do you think that anyone has actually stretched a tape measure from the earth to the sun, or drilled a hole through the center of the earth to measure its diameter? Of course not. These distances are measured *indirectly*. We measure certain lengths and angles that are within our reach. Then we *calculate* the lengths we are interested in. In order to do this, we may use the relations between the parts of a triangle.

We can also use indirect measurements in problems closer to our everyday experiences. Suppose on a sunny day we wish to find the height of a building. We can measure the length of its shadow, which turns out to be 40 feet. Now we ask the help of a friend who is 6 feet tall. We find that the length of his shadow is 8 feet. Thus, his height is six-eighths, or three-fourths, of the length of his shadow. We can write the ratio like this:

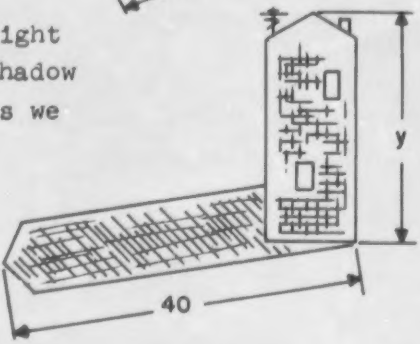


$$\frac{\text{his height}}{\text{length of shadow}} = \frac{6}{8} = \frac{3}{4}$$

It would seem that the ratio of the height of the building to the length of the shadow of the building would also be $\frac{3}{4}$. Thus we obtain the following proportion:

$$\frac{y}{40} = \frac{3}{4}.$$

Using the multiplication property of equality,

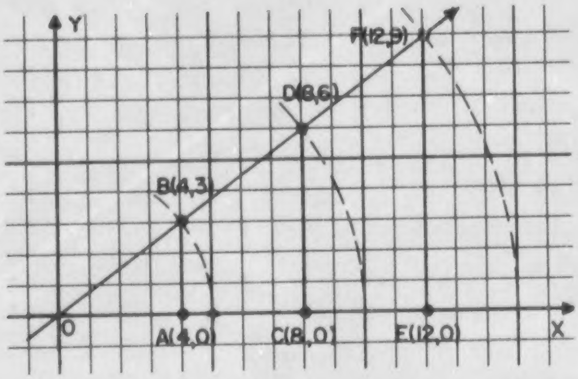


$$40\left(\frac{y}{40}\right) = 40\left(\frac{3}{4}\right)$$

thus, $y = 30$. The height of the building is 30 feet.

Class Exercises 9-1a

In the following drawing, the ordered pair (4, 0) is named with the letter A, (4, 3) is named with B, and so on. $\overrightarrow{OB}$ passes through points D and F. The length of one side of one of the small squares will represent one standard unit of length. Use the figure in answering the questions as indicated.



1. AB is 3, and OA is 4. (Recall that AB with no symbol above it means "the measure of segment $\overline{AB}$.") Therefore the ratio of AB to OA is $\frac{AB}{OA} = \frac{3}{4}$.
   - (a) What is the ratio of CD to OC?
   - (b) What is the ratio of EF to OE?

2. The dotted, curved lines through B, D, and F were drawn with a compass. Thus we can determine that OB is 5, OD is 10, and so on.
   - (a) What is the ratio of AB to OB?
   - (b) What is the ratio of CD to OD?

3. (a) Find the value of $\frac{OA}{OB}$.

   (b) Find $\frac{OC}{OD}$.

4. Copy and complete the table at the right. The completed table will show ratios from the previous drawing.

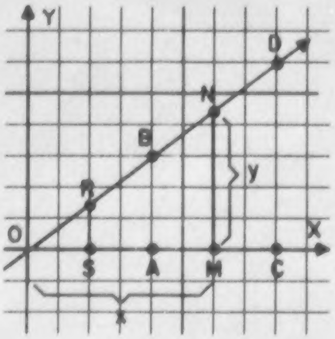
1	2	3
$\frac{AB}{OA} = \frac{3}{4}$	$\frac{AB}{OB} = ?$	$\frac{OA}{OB} = ?$
$\frac{CD}{OC} = ?$	$\frac{CD}{OD} = ?$	$\frac{OC}{OD} = ?$
$\frac{EF}{OE} = ?$	$\frac{EF}{OF} = ?$	$\frac{OE}{OF} = ?$

5. (a) Compare the ratios in Column 1. You should find that these ratios are equal. Thus $\frac{AB}{OA} = \frac{CD}{OC} = \frac{EF}{OE} = \frac{GH}{OG} = \frac{?}{?}$.

   (b) Compare the ratios in Column 2. Are they equal? Which of the ratios is in the simplest form?

   (c) Which of the ratios is in the simplest form in Column 3? Are the ratios in this column equal?

6. A part of the previous drawing is shown at the right. We select a point N on $\overrightarrow{OB}$. A line through N perpendicular to the X-axis intersects the X-axis at M. The length of $\overline{MN}$ is y. The length of $\overline{OM}$ is x.



   (a) y is about $4\frac{1}{2}$ or 4.5. What is x?

   (b) $\frac{MN}{OM} = \frac{y}{x} = \frac{?}{?}$.

7. Suppose we select any point R on $\overrightarrow{OB}$. A line through R is perpendicular to the X-axis and intersects the X-axis at S.

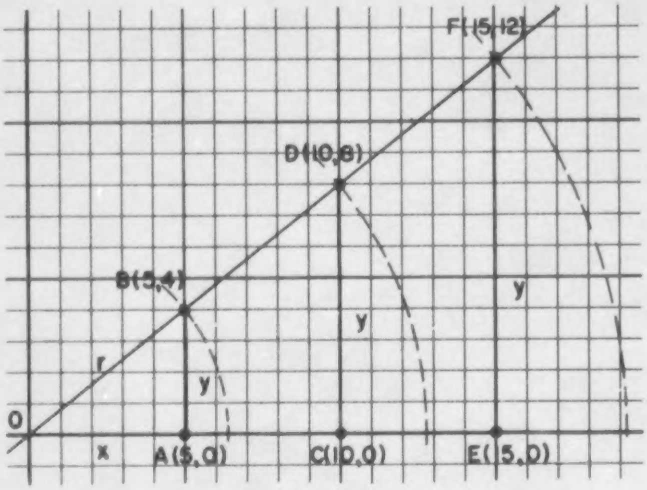
   (a) What kind of a triangle is determined by the line segments joining O, S, and R?

   (b) Does it appear that the ratio of the "y" side to the "x" side of such a triangle will always be $\frac{3}{4}$ if we use this particular ray?

8. (a) In $\triangle OMN$, $\overline{ON}$ is the longest side. What name is given to the longest side of a right triangle?

(b) Since $\triangle OMN$ is a right triangle, the Pythagorean Theorem may be used to find the length of $\overline{ON}$. Find ON using this theorem.

9. In the drawing at the right, note that the angle that $\overrightarrow{OB}$ makes with the X-axis is different from the angle in the drawing for the first problem. For $\triangle OAB$ in this drawing, AB is y, OA is x, and OB is r. Similarly in $\triangle OCD$ and $\triangle OEF$, the measures of the lengths of the vertical sides are y, the horizontal sides x, and the third sides r. For the three triangles we cannot obtain the exact measure of r, the hypotenuse. For these triangles, r is about 6.4, 12.8, and 19.2. Using the information given above, complete the table:



For $\triangle OAB$	$\frac{y}{x} =$	$\frac{y}{r} =$	$\frac{x}{r} =$
For $\triangle OCD$	$\frac{y}{x} =$	$\frac{y}{r} =$	$\frac{x}{r} =$
For $\triangle OEF$	$\frac{y}{x} =$	$\frac{y}{r} =$	$\frac{x}{r} =$

10. (a) What kind of triangles are $\triangle OAB$, $\triangle OCD$, and $\triangle OEF$?

(b) Explain why the measure of the angle at O is the same for each of the triangles.

(c) Is $\frac{y}{x}$ equal to the same number for each of the triangles?

(d) For each triangle, does $\frac{y}{r}$ equal the same number?

(e) Is $\frac{x}{r}$ equal to the same number for each of the triangles?

11. Use the Theorem of Pythagoras to verify that the measure of the length of the hypotenuse for each triangle is about 6.4, 12.8, and 19.2.

Exercises 9-1a

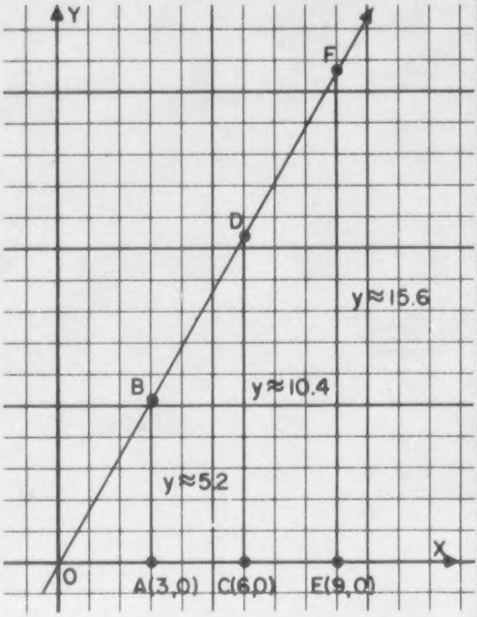
1. On a sheet of graph paper draw a figure similar to those in the class exercises. The coordinates of point A are (3, 0) and the coordinates of point B are (3, 4).

   (a) Find the values of $\frac{y}{x}$, $\frac{y}{r}$, $\frac{x}{r}$ for Δ OAB.

   (b) The coordinates of point D are (6, 8) and the coordinates of point C are (6, 0). Find the values of $\frac{y}{x}$, $\frac{y}{r}$, $\frac{x}{r}$ for Δ OCD. Compare your answers for (b) with those for (a).

   (c) Select two different points on $\overrightarrow{OB}$ and find the values of the three ratios. Compare your answers for (c) with those for (a) and (b).

2. In the drawing at the right, the ray through O is drawn so that it forms with the X-axis an angle of 60°. Lines through B, D, and F are drawn perpendicular to the X-axis, intersecting the X-axis at points A, C, and E with coordinates as indicated. The approximate lengths of y are shown in the drawing. For the three triangles, r, the hypotenuse, has a measure 6, 12, and 18 respectively.



(a) Find the values of $\frac{y}{x}$, $\frac{y}{r}$, and $\frac{x}{r}$ for each triangle and complete a table similar to that in the class exercise.

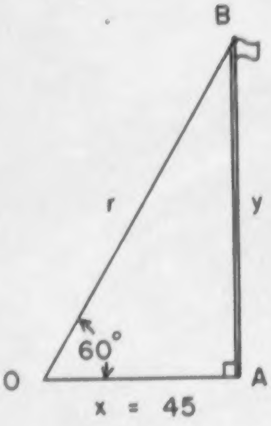
(b) Compare the values of $\frac{y}{x}$ for each triangle. Are they equal?

(c) Are the values of $\frac{y}{r}$ for each triangle equal?

(d) Are the values of $\frac{x}{r}$ for each triangle equal?

(e) Select any other point of $\overrightarrow{OB}$. Find the lengths of x, y, and r. Then determine the three ratios for this new triangle and compare them with the ratios you found in the first part of this problem.

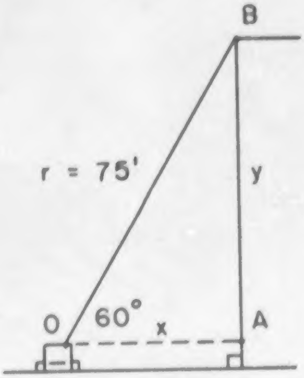
3. In the drawing at the right, ΔAOB is a right triangle. $\angle AOB$ is a 60° angle as in problem 2.



(a) From problem 2, what is the value of $\frac{y}{x}$?

(b) Let T represent the value of $\frac{y}{x}$. Then, $\frac{y}{x} = T$. Write $\frac{y}{x} = T$, replacing "T" with your answer from (a) and replacing x with 45. Find the approximate height of the flagpole, y.

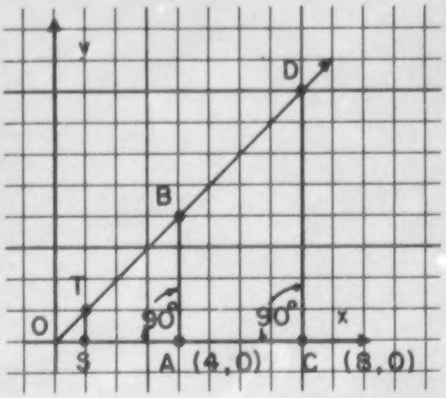
4. The drawing at the right shows a ladder extending from a fire truck to the top of a building. ΔAOB is a right triangle, and $\angle AOB$ is a 60° angle.



(a) From problem 2, what is the value of $\frac{y}{r}$?

(b) Replace "K" in $\frac{y}{r} = K$ with your answer from (a), and replace r with 75. Disregard the height of the fire truck and find the approximate length of y.

5. Use the drawing at the right to answer the following questions.



(a) $\overline{AO} \cong \overline{AB}$ and $\overline{CO} \cong \overline{CD}$. What kind of triangles are Δ OAB and Δ OCD?

(b) Find the measure of $\angle$ AOB without using a protractor.

(c) What is $\frac{y}{x}$ for each of the triangles?

(d) Select any point on $\overrightarrow{OB}$ and find $\frac{y}{x}$. Then compare your answer with that in (c).

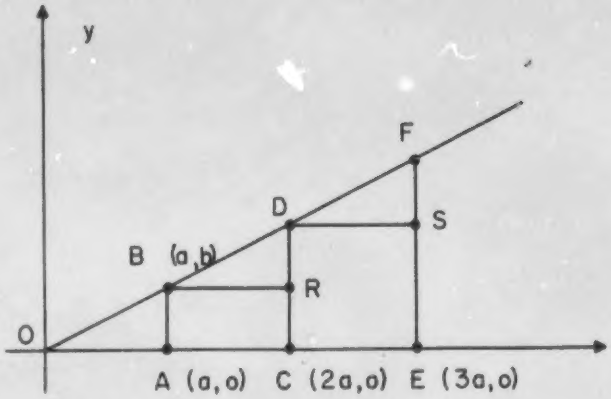
(e) Is the value of $\frac{y}{x}$ the same when the angle the ray makes with the X-axis is 30° and when the angle is 45°?

6. Using the drawing for Problem 5, show that in Δ OST, the value of $\frac{y}{r}$ may be written as $\frac{1}{\sqrt{2}}$.

From the answers to the questions in the previous exercises it would appear that the ratios $\frac{y}{x}$, $\frac{y}{r}$, and $\frac{x}{r}$ depend on which ray is selected through the origin. If the angle the ray makes with the X-axis is 60°, $\frac{y}{x}$ is about 1.73 for any point on the ray. If the angle the ray makes with the X-axis is 45°, $\frac{y}{x}$ is 1.00 for any point on the ray. Thus, the ratio depends on the angle the ray makes with the X-axis and not on the point chosen on the ray. You will now consider why this must be so.

Class Exercises 9-1b

1. In the drawing, the coordinates of Point A are (a, 0) where a is some positive number. The coordinates of Point B are (a, b) where b is some positive number. The angles at A, C, and E are right angles. $\overline{BR}$ and $\overline{DS}$ are parallel to the X-axis.



   (a) The measure of the length of $\overline{OA}$ is a. What other segments are congruent to $\overline{OA}$?

   (b) Show that Δ OAB, Δ BRD, and Δ DSF are congruent.

   (c) $\overline{AB}$ is b units long. What other segments are congruent to $\overline{AB}$?

   (d) What is the measure of the length of $\overline{CD}$?

   (e) What is the measure of the length of $\overline{EF}$?

2. Copy and complete the following table. Note that c is the length of the hypotenuse for each triangle.

For Δ OAB	$\frac{y}{x} = \frac{b}{a}$	$\frac{y}{r} = \frac{b}{c}$	$\frac{x}{r} = ?$
For Δ OCD	$\frac{y}{x} = \frac{2b}{2a}$	$\frac{y}{r} = ?$	$\frac{x}{r} = ?$
For Δ OEF	$\frac{y}{x} = \frac{3b}{3a}$	$\frac{y}{r} = ?$	$\frac{x}{r} = ?$

3. (a) Does the ratio $\frac{1}{3} = \frac{2 \cdot 1}{2 \cdot 3}$? Explain.

   (b) Does the ratio $\frac{a}{b} = \frac{2a}{2b}$? Does $\frac{2a}{2b} = \frac{3a}{3b}$?

   (c) Does $\frac{y}{x}$ have the same value for each of the triangles? What about the value of $\frac{y}{r}$? Of $\frac{x}{r}$?

4. Select a point on $\overrightarrow{OB}$ having the coordinates (x, y), such that $x = 4a$.

   (a) What is the value of y?

   (b) What is the value of the ratio $\frac{y}{x}$?

5. Select a point on $\overrightarrow{OB}$ having the coordinates (x, y), such that $x = \frac{1}{2}a$.

   (a) What is the value of y?

   (b) What is the value of the ratio $\frac{y}{x}$?

It would appear that for every point having coordinates (x, y) on a particular ray $\overrightarrow{OB}$, all $\frac{y}{x}$ represent the same number. Similarly, all $\frac{y}{r}$ represent the same number, and also, all $\frac{x}{r}$ represent the same number. Thus, the ratios are determined only by the angle formed by the ray from the origin through the point (x, y) and the positive X-axis.

Exercises 9-1b

1. Use the drawing at the right to find the answers for the following questions:

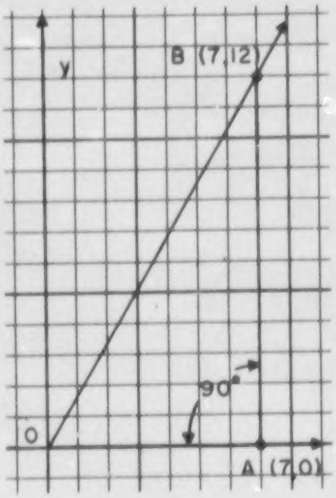
   (a) Determine the value of $\frac{y}{x}$.

   (b) Use the Pythagorean Theorem to determine the length of $\overline{OB}$.

   (c) Find the value of $\frac{y}{r}$.

   (d) Find the value of $\frac{x}{r}$.

   (e) Using a protractor, find the measurement of $\angle$ AOB.

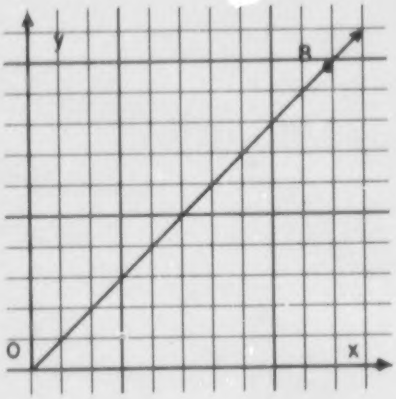


2. In the drawing at the right $\overline{OB}$ is congruent to the $\overline{OB}$ in the drawing for Problem 1. Note that the angles formed by the rays with the X-axis are not the same.

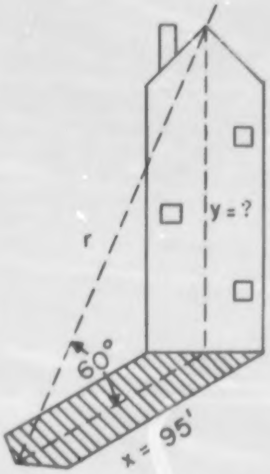
   (a) Determine the value of $\frac{y}{x}$.

   (b) Use the Pythagorean Theorem to determine the length of $\overline{OB}$.

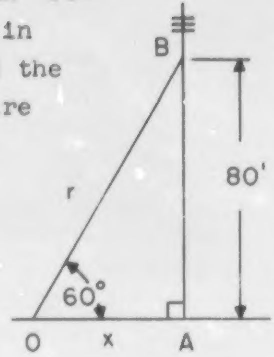
   (c) Find the value of $\frac{y}{r}$ and $\frac{x}{r}$ and compare the results with problem 1(c) and 1(d). Are they the same?



3. In the drawing at the right the height of the building is y. When the angle of the sun's rays with the horizontal is 60°, the length of the shadow of the building, x, is 95 feet. Find y if $\frac{y}{x}$ is about 1.73.

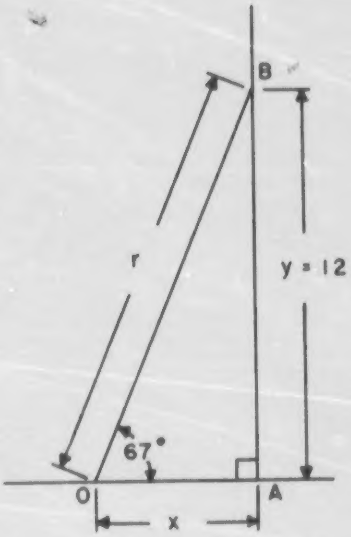


4. A television antenna is mounted on an 80-foot pole. $\frac{y}{r} \approx .87$. Replace "y" in the open sentence with 80 and find the approximate length of the support wire from O to B.



5. In the drawing for Problem 4, when the measurement of $\angle AOB$ is 60°, what is the measurement of $\angle ABO$?

6. In the drawing at the right $\overline{OB}$ represents a ladder leaning against the wall. The top of the ladder reaches a point 12 feet above the ground. Use the following values in answering the questions: $\frac{y}{x} \approx 2.36$; $\frac{y}{r} \approx .92$.

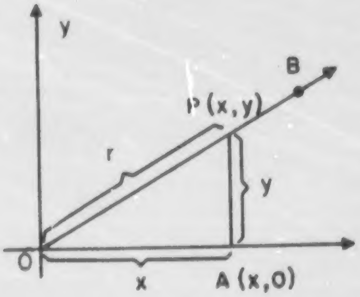


(a) How far is the foot of the ladder from the base of the wall?

(b) How long is the ladder?

(c) Assume the foot of the ladder is five feet away from the base of the building. Use the Pythagorean Theorem to check your answer for (b)

9-2. Trigonometric Ratios



In the drawing, consider a particular ray $\overrightarrow{OB}$ through the origin. On this ray, for any point P, whose coordinates are (x, y), a right triangle is determined with the right angle at a point on the X-axis. In the previous section you learned that $\frac{y}{x}$ is the same for all points on $\overrightarrow{OB}$. In other words, $\frac{y}{x}$ depends only on the angle AOB and not on the particular point P chosen. We call this ratio the tangent of the angle AOB. The tangent of angle AOB is abbreviated tan $\angle$ AOB.

$$\tan \angle AOB = \frac{y}{x} = \frac{AP}{OA} = \frac{\text{length of the opposite side}}{\text{length of the adjacent side}}.$$

You also learned that the value of $\frac{y}{r}$ is the same for all points on the ray $\overrightarrow{OB}$, where r is the measure of the length of OP, that is, the hypotenuse of ΔAOP. We call this ratio the sine of angle AOB. The sine of angle AOB is abbreviated $\sin \angle AOB$.

$$\sin \angle AOB = \frac{y}{r} = \frac{AP}{OP} = \frac{\text{length of the opposite side}}{\text{length of the hypotenuse}}.$$

Finally, you learned that the value of $\frac{x}{r}$ is the same for all points on the ray $\overrightarrow{OB}$. We call this ratio the cosine of the angle AOB, and we abbreviate this $\cos \angle AOB$.

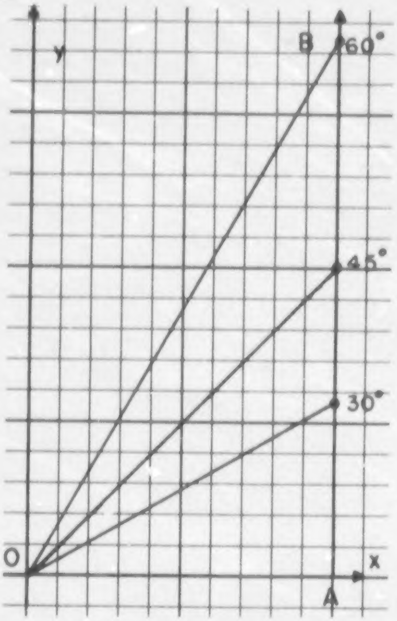
$$\cos \angle AOB = \frac{x}{r} = \frac{OA}{OP} = \frac{\text{length of the adjacent side}}{\text{length of the hypotenuse}}.$$

In Exercises 9-1a, we found the following ratios for an angle of 60°:

$$\tan 60^\circ \approx 1.73; \quad \sin 60^\circ \approx 0.87; \quad \cos 60^\circ = 0.50.$$

Class Exercises 9-2

1. In the figure at the right, the line through A is perpendicular to the X-axis. Rays through the origin intersect the line. Call the point of each intersection B.



   (a) For each of the triangles, what is the measure of the length of $\overline{OA}$?

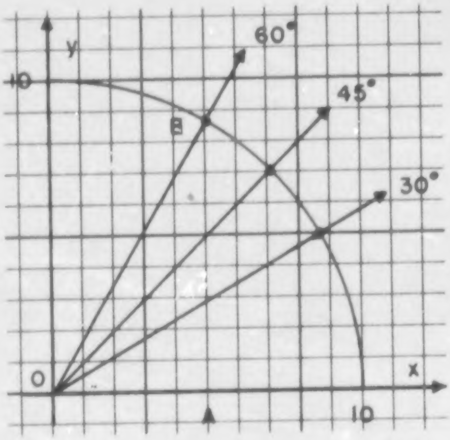
   (b) For the ray marked 60°, what is the measure of the length of $\overline{AB}$?

   (c) Use your answers for (a) and (b) to calculate the tangent of $\angle BOA$. (Calculate your results to 2 decimal places.)

[sec. 9-2]

(d) Similarly, calculate the tangents of 45° and 30°.

2. In the drawing at the right, the radius of the arc is 10 units. Let B be the point where each ray through the origin intersects the circle. A line through B is perpendicular to the X-axis. For each line the point of intersection is A. Calculate your results to two decimal places.



(a) For each of the triangles, what is the measure of the length of $\overline{OB}$?

(b) For the triangle determined by the ray marked 60° what is the measure of the length of $\overline{AB}$?

(c) Use your answers for (a) and (b) and calculate the sine of $\angle BOA$.

(d) Similarly, calculate the sine of 45° and of 30°.

(e) For the triangle determined by the ray marked 60°, what is the measure of the length of $\overline{OA}$?

(f) Use your answers for (a) and (e) and calculate the cosine of $\angle BOA$. (Calculate your results to 2 decimal places.)

(g) Similarly, calculate the cosine for the rays marked 45° and 30°.

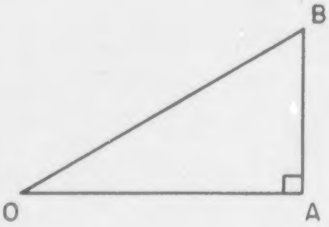
3. (a) How does the sine of a 30° angle compare with the cosine of a 60° angle?

(b) Compare the sine of a 45° angle with the cosine of a 45° angle.

4. Using your answers for Problems 1 and 2, copy and complete the following table. Keep a copy of the table, since it will be needed in Exercises 9-2.

$m(\angle BOA)$	$\sin \angle BOA$	$\tan \angle BOA$	
30			60
45			45
60			30
	$\cos \angle BOA$	(Use the column at the right for the measure of the angle when using cosine of the angle.)	

5. Consider the right triangle in the drawing. $\angle BOA$ has a measurement of 30°, and the measurement of $\overline{OA}$ is 6 feet.



(a) Which one of the trigonometric ratios involves the opposite and adjacent sides?

(b) From your table, what is the measure of the ratio $\frac{AB}{OA}$ for $\angle BOA$?

(c) Let y represent the measure of the length of $\overline{AB}$. Then,

$$\frac{y}{6} = \tan \angle BOA.$$

Solve to find the measure of the length of $\overline{AB}$.

6. When we say that "$\tan (\angle AOB) = \frac{y}{x}$," we also mean that "$\frac{y}{x} = \tan (\angle AOB)$." That is, we can go from left to right or from right to left with our definition of the trigonometric ratios. Or, you might say that the ratios are used "forwards and backwards." You should learn them in both ways. In each case below, state which trigonometric ratio is defined

in terms of the two stated sides of a right triangle:

(a) hypotenuse and opposite side.

(b) adjacent side and hypotenuse.

(c) adjacent side and opposite side.

(d) hypotenuse and adjacent side.

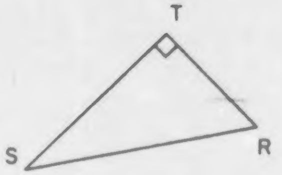
7. For this triangle:

(a) For $\angle$ RST, the opposite side is ____.

(b) For $\angle$ RST, the adjacent side is ____.

(c) For $\angle$ SRT, the opposite side is ____.

(d) For $\angle$ SRT, the adjacent side is ____.



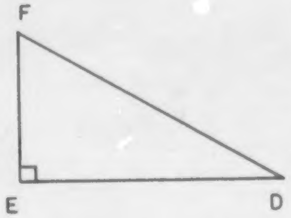
8. For this triangle:

(a) The side opposite $\angle$ EDF is ____.

(b) The side adjacent to $\angle$ EDF is ____.

(c) The side opposite $\angle$ DFE is ____.

(d) The side adjacent to $\angle$ DFE is ____.

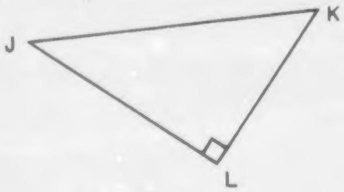


9. For the triangle, first name the opposite side for the given angle, and then name the side adjacent to the given angle.

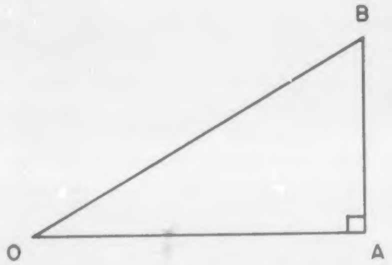
(a) For $\angle$ LJK.

(b) For $\angle$ JKL.

(c) The hypotenuse is ____.



10. For the right triangle OAB assume that you are given the trigonometric ratios for the tangent of $\angle$ AOB, sine of $\angle$ AOB, and cosine of $\angle$ AOB. Which trigonometric ratio would you use to find each of the remaining two sides of the



triangles when you know the following:

(a) OA = 7.

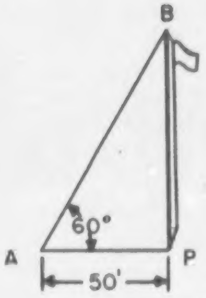
(b) OB = 6.

(c) AB = 2.

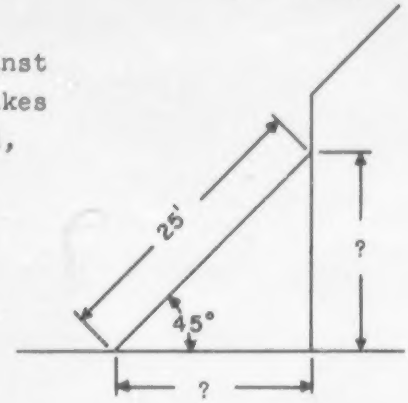
11. For the previous problem, answer each of the questions for $\angle$ ABO instead of $\angle$ AOB.

Exercises 9-2

1. The point A is 50 feet from the foot of a flagpole. Find the height of the flagpole if m($\angle$ PAB) = 60°. ($\angle$ PAB is called the angle of elevation of the top of the pole.)

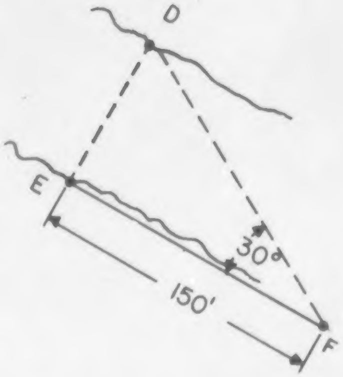


2. A 25-foot ladder is placed against the side of a building. If it makes an angle of 45° with the ground, at what height does it touch the building?



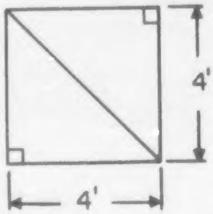
3. Use the drawing for the previous problem to determine how far from the foot of the building the foot of the ladder is placed.

4. To find the measure of the width of a river two boys set up stakes at E and F, using a tree on the opposite bank for D. DEF is a right angle. The measure of angle DFE is 30. If the distance from E to F is 150 feet, how wide is the river?



5. (a) The side of a square has a measurement of 4 feet. Find the length of the diagonal using one of the trigonometric ratios.

   (b) Check your answer for (a) using the Pythagorean Theorem.



6. Find the ratios: $\frac{\sin 60^\circ}{\sin 30^\circ}$ and $\frac{\tan 60^\circ}{\tan 30^\circ}$

   State in your own words why you think these ratios are not 2.

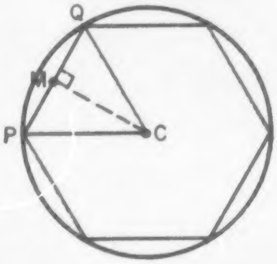
7. A regular hexagon is inscribed in a circle of radius 10 inches.

   (a) What is the measure of $\angle$ PCQ?

   (b) What is the measure of $\angle$ CPQ?

   (c) Find CM.

   (d) Find PQ.



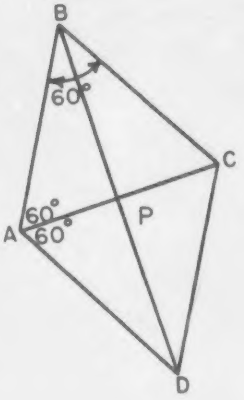
8. The figure ABCD consists of the union of two equilateral triangles ABC and ACD. The sides of the triangles are 10 inches long.

   (a) What is the measure of $\angle ABD$?

   (b) What is the measure of $\angle DBC$?

   (c) Why is $\overline{BD} \perp \overline{AC}$?

   (d) Find the approximate measure of the length of $\overline{BD}$.

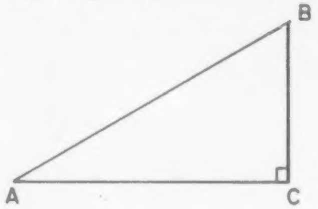


9-3. Reading a Table

You used a table of square roots in Chapter 4, and in Class Exercises 9-2 you made a table of the values of the trigonometric ratios for the following three angles of measurement: 30°, 45°, and 60°. Perhaps you noticed that $\sin 30^\circ = \cos 60^\circ$. Do you think this might be true for other angles?

You may have noticed that for any right triangle, the sum of the measures of the two smaller angles is 90. The sum of 30° and 60° is 90°. Similarly, the sum of 45° and 45° is 90°. Two angles, the sum of whose measures is 90, are called *complementary angles*. A 30° angle and a 60° angle are complementary angles. Similarly, a 20° angle and a 70° angle are complementary angles. In fact, the two non-right angles of a right triangle will always be complementary angles.

Look at the following figure, and refer to the trigonometric ratios below:



$$\sin \angle CAB = \frac{\text{length of the opposite side}}{\text{length of the hypotenuse}} = \frac{CB}{AB}$$

$$\cos \angle CBA = \frac{\text{length of the adjacent side}}{\text{length of the hypotenuse}} = \frac{CB}{AB}$$

We have shown that the $\sin \angle CAB = \cos \angle CBA$. Angles CAB and CBA are complementary angles. Thus it would appear that for any pair of complementary angles, the sine of one of the angles would be equal to the cosine of the other angle. Does $\sin \angle CBA = \cos \angle CAB$ in the drawing? Note that both of these ratios are the same, $\frac{AC}{AB}$. Thus, it is no accident that $\sin 30^\circ = \cos 60^\circ$. It would also be true that $\sin 20^\circ$ is equal to $\cos 70^\circ$, $\cos 10^\circ = \sin 80^\circ$, and so on.

We used this property to shorten the table we made for three angles. In the same way, this property is used in the table of trigonometric ratios on page 368. In this table, the measurements for angles from 1° to 45° are listed on the left. The measurements for angles from 45° to 89° are listed on the right, reading from the bottom of the page toward the top. The ratio headings are shown at the top of the page for all angles from 1° to 45°. For angles 45° to 89° the ratio headings are shown at the bottom of the page.

To find $\sin 20^\circ$ in the table, first look for 20° in the column on the left since $20 < 45$. Next, find the column headed "sine" at the top of the page. The number in the sine column and the 20° row is .3420. Thus, $\sin 20^\circ \approx 0.3420$ correct to four decimal places. Notice that the approximation sign is used indicating that we know only that the decimal is correct to four places.

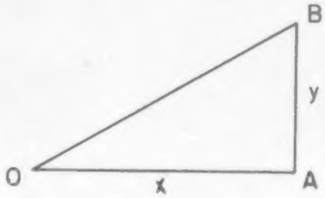
To find $\cos 70^\circ$, look for the angle measurement in the right-hand column since $70 > 45$. Because you are using the right-hand column, look for the ratio headings at the bottom of the table. In the column headed "cosine" at the bottom of the page we find 0.3420 in the row for 70°. Thus, $\cos 70^\circ \approx 0.3420$ to the nearest ten-thousandths. Notice that this is the same as $\sin 20^\circ$. Find $\sin 70^\circ$. It is about 0.9397.

Notice that there is another ratio listed in the table, that of cotangent. Just as we call the sine of the complement of an angle the cosine of that angle, so we call the tangent of the complement of an angle the cotangent of the angle. For the triangle at the right,

$$\tan \angle AOB = \frac{y}{x} = \cot \angle ABO$$

and,

$$\tan \angle ABO = \frac{x}{y} = \cot \angle AOB.$$



Notice that the "co" in cosine and cotangent are suggested by the "co" in "complementary." Angles AOB and ABO in the previous drawing are complementary angles.

You can find more complete and more accurate tables of trigonometric ratios in a library. People who use mathematics in their work usually own books containing various sets of mathematical tables.

TRIGONOMETRIC RATIOS

Angle	Sine	Tangent	Cotangent	Cosine	
1°	.0175	.0175	57.290	.9998	89°
2°	.0349	.0349	28.636	.9994	88°
3°	.0523	.0524	19.081	.9986	87°
4°	.0698	.0699	14.301	.9976	86°
5°	.0872	.0875	11.430	.9962	85°
6°	.1045	.1051	9.5144	.9945	84°
7°	.1219	.1228	8.1443	.9925	83°
8°	.1392	.1405	7.1154	.9903	82°
9°	.1564	.1584	6.3138	.9877	81°
10°	.1736	.1763	5.6713	.9848	80°
11°	.1908	.1944	5.1446	.9816	79°
12°	.2079	.2126	4.7046	.9781	78°
13°	.2250	.2309	4.3315	.9744	77°
14°	.2419	.2493	4.0108	.9703	76°
15°	.2588	.2679	3.7321	.9659	75°
16°	.2756	.2867	3.4874	.9613	74°
17°	.2924	.3057	3.2709	.9563	73°
18°	.3090	.3249	3.0777	.9511	72°
19°	.3256	.3443	2.9042	.9455	71°
20°	.3420	.3640	2.7475	.9397	70°
21°	.3584	.3839	2.6051	.9336	69°
22°	.3746	.4040	2.4751	.9272	68°
23°	.3907	.4245	2.3559	.9205	67°
24°	.4067	.4452	2.2460	.9135	66°
25°	.4226	.4663	2.1445	.9063	65°
26°	.4384	.4877	2.0503	.8988	64°
27°	.4540	.5095	1.9626	.8910	63°
28°	.4695	.5317	1.8807	.8829	62°
29°	.4848	.5543	1.8040	.8746	61°
30°	.5000	.5774	1.7321	.8660	60°
31°	.5150	.6009	1.6643	.8572	59°
32°	.5299	.6249	1.6003	.8480	58°
33°	.5446	.6494	1.5399	.8387	57°
34°	.5592	.6745	1.4826	.8290	56°
35°	.5736	.7002	1.4281	.8192	55°
36°	.5878	.7265	1.3764	.8090	54°
37°	.6018	.7536	1.3270	.7986	53°
38°	.6157	.7813	1.2799	.7880	52°
39°	.6293	.8098	1.2349	.7771	51°
40°	.6428	.8391	1.1918	.7660	50°
41°	.6561	.8693	1.1504	.7547	49°
42°	.6691	.9004	1.1106	.7431	48°
43°	.6820	.9325	1.0724	.7314	47°
44°	.6947	.9657	1.0355	.7193	46°
45°	.7071	1.0000	1.0000	.7071	45°
	Cosine	Cotangent	Tangent	Sine	Angle

Exercises 9-3

1. Use the table to find the following:

(a)	$\sin 10^\circ$	(f)	$\tan 40^\circ$
(b)	$\tan 10^\circ$	(g)	$\tan 50^\circ$
(c)	$\sin 41^\circ$	(h)	$\tan 60^\circ$
(d)	$\sin 63^\circ$	(i)	$\tan 70^\circ$
(e)	$\sin 82^\circ$	(j)	$\sin 88^\circ$

2. Check the statements below by studying the numbers in the table. Do you agree with the statements?
   (a) The sine of an angle in the table is always between 0 and 1.
   (b) The sine of an angle increases with the size of the angle from 1° to 89°.
   (c) The sine of an angle less than 30° is less than $\frac{1}{2}$.
   (d) The differences between consecutive table readings varies throughout the table.
   (e) The difference between the sines of two consecutive angles in the table is greater for smaller consecutive angles than for larger consecutive angles.

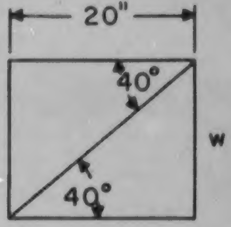
3. Use the tangents of angles given in the tables to answer the following questions:
   (a) Is the tangent of an angle always between 0 and 1?
   (b) Does the tangent of an angle between 1° and 89° increase with the angle?
   (c) Do the differences between consecutive readings in the table vary throughout the table?
   (d) Is the difference between the tangents of two consecutive angles in the table greater for smaller consecutive angles than for larger consecutive angles?

4. Find the following products:

   (a) $100 \cdot (\sin 32^\circ)$ (c) $0.27 \cdot (\sin 73^\circ)$

   (b) $81 \cdot (\tan 48^\circ)$ (d) $0.05 \cdot (\tan 20^\circ)$

5. The diagonal of the rectangle shown at the right makes an angle of 40° with the longest sides. Find the width of the rectangle if its length is 20 inches.

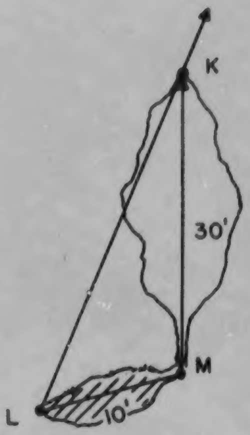


6. Triangle ABC is a right triangle with $\angle$ ACB the right angle. AC = 5, and BC = 4.

   (a) What trigonometric ratio of $\angle$ BAC is $\frac{AC}{BC} = \frac{5}{4}$?

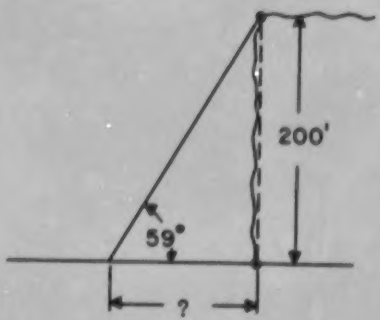
   (b) Use the table of trigonometric ratios to find the approximate measure of $\angle$ BAC.

7. The length of a shadow of a 30-foot tree is 10 feet. What is the approximate measure of the angle of elevation ($\angle$ KLM) of the sun? (Recall that the angle of elevation of an object from a point L is the angle between a horizontal line through L and the line through L and the given object. In this case the given object is the sun.)



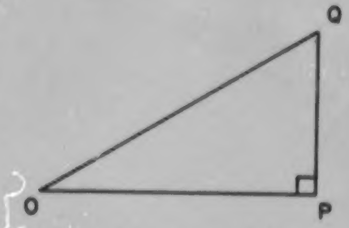
8. Suppose in the previous problem, the length of the shadow had been 20 feet instead of 10 feet. Would your answer be one-half of the previous answer? If so, why? If not, what would the approximate measure of the angle be?

9. An observer sees that the angle of elevation from where he stands to the top of a cliff is 59°. If the cliff is 200 feet high, find, to the nearest foot, the distance from the observer to the foot of the cliff.



10. The tangent of the complement of an angle has been called the cotangent of the angle. In the drawing,

$$\cot \angle POQ = \tan \angle PQO$$



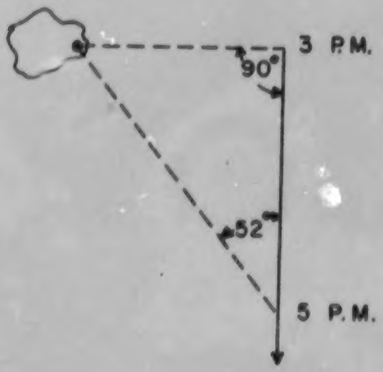
(a) Show that

$$\cot \angle POQ = \frac{1}{\tan \angle POQ}.$$

this would be a different way to define the cotangent: the cotangent of an angle is the reciprocal of the tangent of the angle.

(b) Is the sine of an angle the reciprocal of the cosine of the angle? Explain your answer using the drawing.

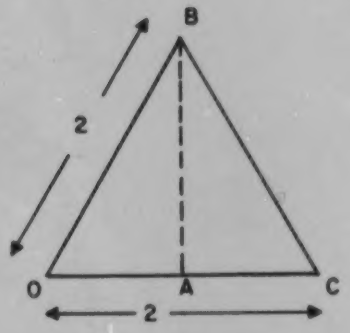
11. The navigator on a ship sailing due south observes a lighthouse due west at 3 p.m. At 5 p.m., the lighthouse is 52° west of north. The ship is moving at a speed of 15 miles per hour.



(a) How far from the lighthouse was the ship at 3 p.m.?

(b) How far from the lighthouse was the ship at 5 p.m.?
(Compute your answer to the nearest tenth of a mile.)

12. Triangle OBC at the right is an equilateral triangle. Each side of the triangle has a measure of 2.



(a) What is the measurement of angles BOC, OBC, and BCO? Explain your answer.

(b) The vertex B is joined to the midpoint A of $\overline{OC}$. Show that $\Delta OAB \cong \Delta CAB$.

(c) Show that ΔOAB is a right triangle.

(d) Find the measurement of $\angle OBA$.

(e) Find the length of $\overline{AB}$ using a trigonometric ratio.

13. Refer to the drawing from Problem 12 $m(\angle COB) = 60^\circ$. The measure of $\overline{OB} = 2$, and the measure of $\overline{OA} = 1$.

(a) $\cos \angle AOB = \cos 60^\circ = \frac{OA}{OB} = \frac{?}{?}$.

(b) In order to find the $\sin \angle AOB$ and $\tan \angle AOB$ we must find the measure of $\overline{AB}$ which we shall call y. By the Pythagorean Theorem,

$$(OA)^2 + (AB)^2 = (OB)^2$$

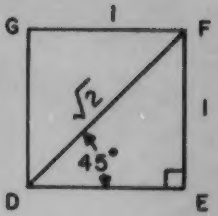
or, $1^2 + y^2 = 2^2$

thus, $y^2 = ?$

and, $y = ?$

(c) Use your results for part (b) to find $\sin 60^\circ$ and $\tan 60^\circ$ and check your results with the values given in the table.

14. The square shown at the right has a measure of 1 for each side.



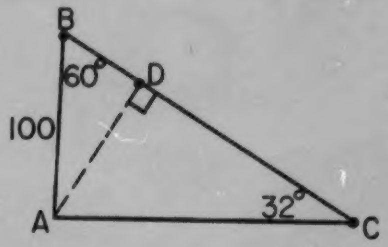
(a) Verify that the measure of DF is $\sqrt{2}$.

(b) If $DF = \sqrt{2}$, then $\sin 45^\circ = \frac{1}{\sqrt{2}}$. You can determine a decimal expression for $\frac{1}{\sqrt{2}}$ by dividing 1 by 1.4142, but this is a tedious computation. Recall that any number divided by itself (except for 0) is 1. Thus,

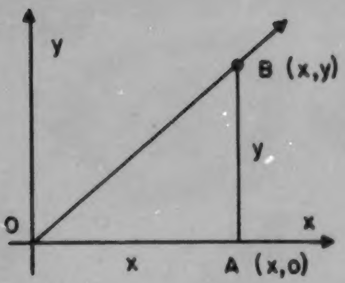
$$\frac{\sqrt{2}}{\sqrt{2}} = 1, \text{ and } \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2}.$$

It is much easier to divide 1.4142 by 2 than to divide 1 by 1.4142. Find the $\sin 45^\circ$ using the above computation and then verify your answer with the table.

*15. In the figure at the right $\angle ABC$ has measure 60, $\angle ACB$ has measure 32, in degrees, and $AB = 100$. Find (a) $m(\overline{AD})$ and (b) $m(\overline{BC})$.



9-4. Slope of a Line



The drawing at the right is similar to those used in Section 9-1. You learned that for a particular ray $\overrightarrow{OB}$ from the origin through the point (x, y), all $\frac{y}{x}$ are equal. For a particular ray then,

$$\frac{y}{x} = m,$$

where m is another name for the ratio. However, you learned that $\frac{y}{x}$ is the tangent of the angle which the ray makes with the X-axis. Thus, m is the tangent of the angle AOB. The number m depends on the particular ray which is chosen and not on any particular point on the ray.

Using the multiplication property of equality,

$$\frac{y}{x}(x) = m(x)$$

and, $$y\left(\frac{x}{x}\right) = mx, \quad \text{but } \frac{x}{x} = 1,$$

so, $$y = mx.$$

Thus, an equivalent equation for $\frac{y}{x} = m$ is $y = mx$. If the tangent of an angle is 2, then,

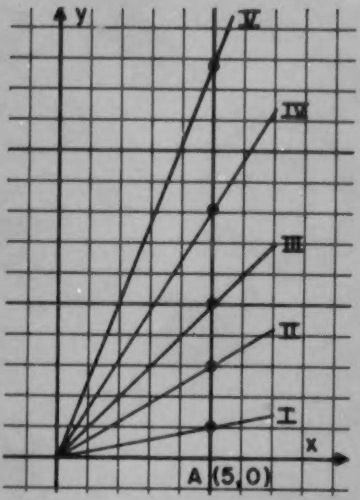
$$\frac{y}{x} = 2 \text{ or } y = 2x.$$

If the tangent of an angle is $\frac{4}{3}$, then,

$$\frac{y}{x} = \frac{4}{3} \text{ or } y = \frac{4}{3}x.$$

Each point on the line which an equation represents will have coordinates which satisfy the equation, and each point whose coordinates satisfy the equation will lie on the line. The number in this equation is called the slope of the line. In the equation $y = 2x$, the "2" is the slope. In the equation $y = \frac{4}{3}x$, the "$\frac{4}{3}$" is the slope, and in the equation $y = mx$, the "m" is the slope. Slope is another name for the tangent ratio of the angle made by a particular ray and the X-axis.

The word "slope," when used with reference to a hill, refers to the steepness of the hill. The slope of a hill is measured by dividing the measure of the change in elevation by the measure of the corresponding horizontal change. In the drawing at the right the rays are named with Roman numerals. The tangents or slopes of the angles the rays make with the X-axis are as follows:



For I, $m = \frac{1}{5}$; for II, $m = \frac{3}{5}$;

for III, $m = \frac{5}{5}$; for IV, $m = \frac{8}{5}$;

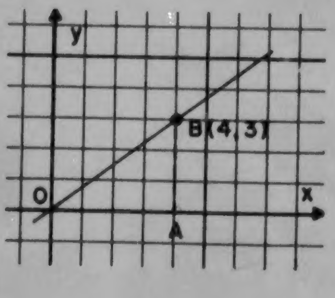
and for V, $m = \frac{13}{5}$. As the y increases, the tangent of the angle, or slope increases. The steeper the ray, the greater the tangent of the angle or the greater the slope.

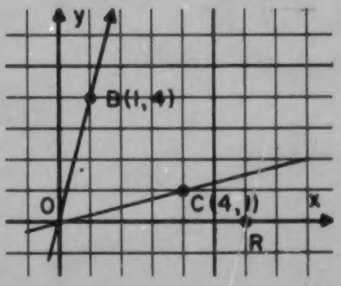
Class Exercises 9-4

1. (a) In the drawing, what is the tangent of the angle the ray makes with the X-axis?

   (b) Use your answer from (a) as a replacement for m in the general equation $y = mx$.

   (c) Your answer for (b) is an equation for the line $\overleftrightarrow{OB}$.



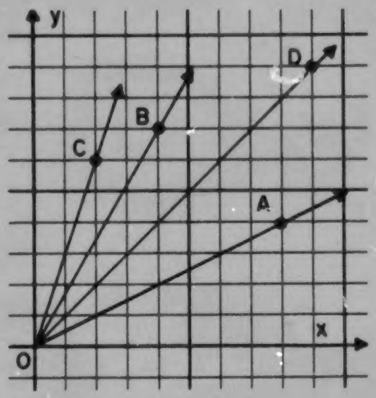


2. (a) Write an equation for the line joining the origin to the point (4, 1).

   (b) Write an equation for the line joining the origin to the point (1, 4).

   (c) What is the tangent of angle COR?

   (d) What is the tangent of angle BOR?



3. Name the line through the origin which is described by each of the following equations. Note that some of the equations are equivalent equations.

   (a) $y = (1)x$

   (b) $y = \frac{1}{2}x$

   (c) $y = 3x$

   (d) $y = \frac{7}{4}x$

   (e) $\frac{y}{x} = \frac{1}{2}$

   (f) $\frac{y}{x} = 1$

   (g) $\frac{y}{x} = \frac{7}{4}$

   (h) $\frac{y}{x} = \frac{3}{1}$

4. Use the drawing for Problem 3 and find the slope for each of the following lines.

   (a) $\overleftrightarrow{OA}$ (b) $\overleftrightarrow{OB}$ (c) $\overleftrightarrow{OC}$ (d) $\overleftrightarrow{OD}$

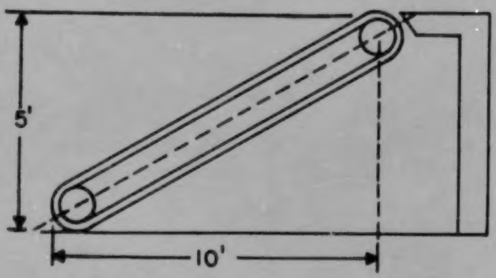
There is another ratio that is sometimes used to designate the steepness of a line. You may have heard of a "road having a 2% grade". This refers to the ratio $\frac{y}{r}$, that is, the sine of the angle which the road makes with the horizontal. A road that rises 2 feet for every 100 feet measured along the road has a 2% grade. For roads which are not very steep, the grade is very close in value to the slope.

Exercises 9-4

1. Find equations for the lines joining the origin to each of the following points.

   (a) (4, 1)
   (b) (3, 1)
   (c) (1, 1)
   (d) (1, 2)
   (e) (1, 5)
   (f) (5, 3)
   (g) (5, 7)
   (h) (6, 2)
   (i) (5, 0)

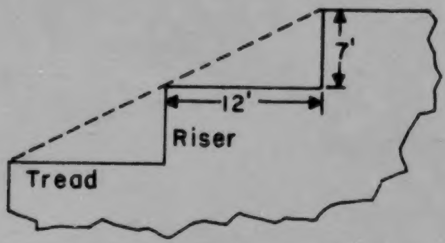
2. A conveyer belt is used to "lift" materials to a loading platform. A box placed on the conveyer belt is lifted 5 feet while it is moved a horizontal distance of 10 feet.



   (a) What is the slope of the path along which the box is carried?

   (b) What is the approximate measurement of the angle of the path along which the box is carried and the horizontal?

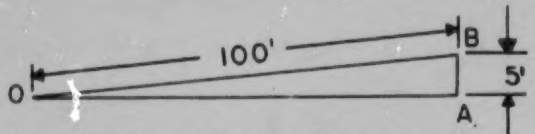
3. The drawing at the right shows a partial side view of a stairway. The slope of the stairway is defined as the slope of the dotted line.



(a) What is the slope of this stairway?

(b) What is the approximate measurement of the angle formed by the dotted line and the horizontal line (tread)?

4. The drawing at the right shows a 5% "grade". That is, in a distance of 100 feet there is an elevation 5 feet. The horizontal distance in this case is about 100 feet. It would actually be a little less than this distance.



(a) What is the sine of angle BOA?

(b) What is the tangent of angle BOA?

(c) What is the approximate measurement of the angle BOA?

(d) Approximately, what is the slope of $\overleftrightarrow{OB}$?

5. Using the drawing for Problem 4, assume the measurement of BOA is 2°.

(a) What is the sin $\angle$ BOA?

(b) What is the tan $\angle$ BOA?

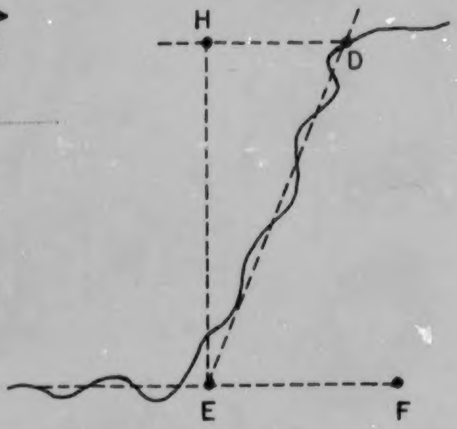
(c) What is the approximate measure of the rise (or elevation) if the horizontal distance is 100 feet?

(d) What is the grade of $\overleftrightarrow{OB}$?

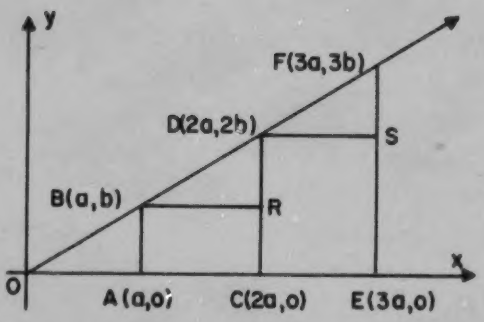
6. In the drawing at the right, $\overleftrightarrow{DE}$ is the approximate slope of the hillside. The tangent of $\angle$ DEF is about 2.05.

(a) What is the approximate measurement of $\angle$ DEF?

(b) What is the approximate vertical height of the hill if the measurement of $\overline{DH}$ is about 500'?

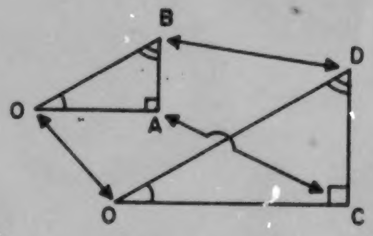


9-5. Similar Triangles



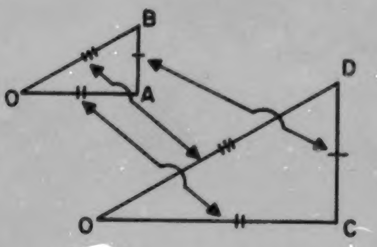
Refer to the drawing at the right which is a copy of the drawing used in Class Exercises 9-1b. Recall that triangles AOB, RBD, and SDF are congruent.

Look at ΔAOB and ΔCOD. These triangles are not congruent, but there are a number of relationships between the two. We can set up a one-to-one correspondence between the vertices of the two triangles as shown at the right. The following angles also correspond:



$\angle AOB \longleftrightarrow \angle COD$;
$\angle OAB \longleftrightarrow \angle OCD$;
$\angle OBA \longleftrightarrow \angle ODC$.

Because ΔAOB and ΔCOD are right triangles, we know that the corresponding angles with vertices at A and at C are congruent. The corresponding angles with vertices at O are the same angle, and thus $\angle AOB \cong \angle COD$. Since $\Delta AOB \cong \Delta RBD$, the corresponding angles with vertices at B and D are congruent. The corresponding angles of these two triangles are congruent.



In view of the correspondence between the vertices, we have the following correspondence between the sides:

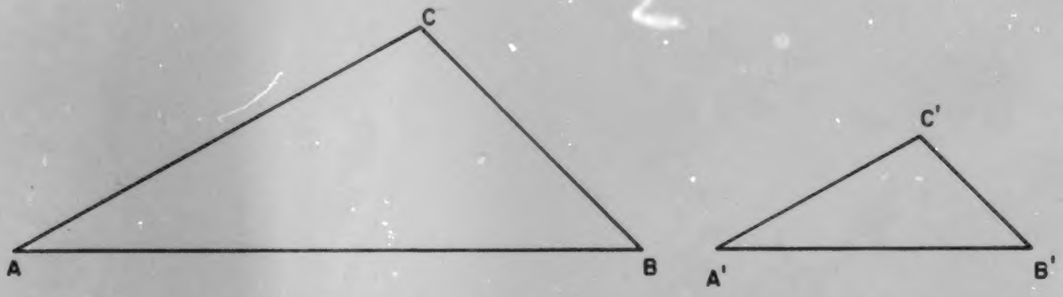
$\overline{OA} \longleftrightarrow \overline{OC}$
$\overline{AB} \longleftrightarrow \overline{CD}$
$\overline{OB} \longleftrightarrow \overline{OD}$

It does not appear that these corresponding sides are congruent. However, we know that the measure of $\overline{OA}$ is a, and we know that the measure of $\overline{OC}$ is 2a. Also, the measure of $\overline{AB}$ is b, and that of $\overline{CD}$ is 2b. With the Pythagorean Theorem it can be shown that the measure of $\overline{OD}$ is two times the measure of $\overline{OB}$. Thus, each of the sides of ΔCOD is twice as long as the corresponding side of ΔAOB. The ratios of the lengths of the corresponding sides are equal:

$$\frac{OA}{OC} = \frac{AB}{CD} = \frac{OB}{OD} = \frac{1}{2}$$

We express this relationship by saying that for these triangles "the corresponding sides are proportional."

So far we have considered only right triangles. In the drawing below are two triangles, ΔABC and $\Delta A'B'C'$.



The corresponding angles are congruent. That is,

$$\angle A \cong \angle A', \quad \angle B \cong \angle B', \quad \text{and} \quad \angle C \cong \angle C'.$$

Each side of ΔABC has twice the length of the corresponding side of $\Delta A'B'C'$. That is

$$AB = 2(A'B'), \quad AC = 2(A'C'), \quad \text{and} \quad BC = 2(B'C').$$

These are examples of what we call "similar triangles."

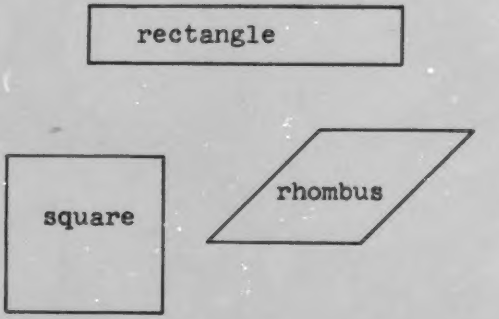
> Definition: Two triangles are said to be similar if there is a one-to-one correspondence between the vertices so that corresponding angles are congruent and the ratios of the measures of corresponding sides are equal (that is, corresponding sides are proportional).

This definition means that any two triangles ABC and A'B'C' are similar if the following two conditions hold:

1. $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, and $\angle C \cong \angle C'$.
2. $\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$.

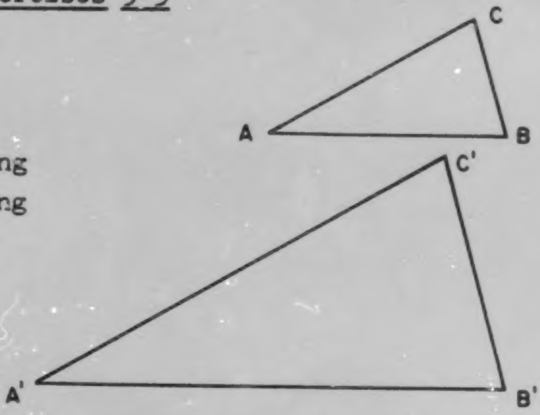
It is possible to prove that if condition 1 holds, then condition 2 must hold, and if condition 2 holds, then condition 1 must hold. (You might try showing that this statement is true.) We will accept the statement here without proof. It follows from the statement that only one of these conditions need be used in defining similar triangles.

The only similar figures we shall consider in this chapter are triangles. For other figures, such as squares and rectangles, both of the conditions listed above are necessary for establishing similarity. For example, the square and rectangle at the right have congruent angles, but the sides are not proportional. The square and the rhombus have congruent sides, but they are not similar.



Class Exercises 9-5

1. ΔABC and $\Delta A'B'C'$ are similar triangles in which A and A', B and B', C and C' are corresponding vertices. Supply the missing information where it is possible. Where it is not possible, explain why.



(a) $m(\angle A) = 30$, $m(\angle B) = 75$, $m(\angle A') = ?$, $m(\angle B') = ?$.

(b) $AB = 3$, $AC = 4$, $A'B' = 6$, $A'C' = ?$

(c) $\frac{AB}{BC} = \frac{2}{3}$, $A'B' = 5$, $B'C' = ?$, $A'C' = ?$.

(d) $\frac{BC}{AC} = \frac{4}{5}$, $A'C' = 3$, $A'B' = ?$, $B'C' = ?$

(e) $m(\angle A) = 30$, $m(\angle B) = 73$, $m(\angle A') = ?$, $m(\angle C') = ?$.

2. Find which of the following are true statements. Give reasons for your answers.

(a) If one acute angle of one right triangle is congruent to an acute angle of another right triangle, then the triangles are similar.

(b) If two sides of one triangle are congruent to two sides of another triangle, then the triangles are similar.

(c) If $\frac{AB}{A'B'} = \frac{AC}{A'C'}$ and $\frac{AB}{A'B'} = \frac{BC}{B'C'}$ then triangles ABC and A'B'C' are similar.

(d) If $\frac{AB}{A'B'} = \frac{AC}{A'C'}$ then triangles ABC and A'B'C' are similar.

(e) If $\frac{AB}{A'B'} = \frac{AC}{A'C'}$ and $\frac{A'B'}{AB} = \frac{B'C'}{BC}$ then triangles ABC and A'B'C' are similar.

3. (a) If the corresponding angles of two quadrilaterals are congruent, must the ratios of the measures of corresponding sides be equal?

(b) If the ratios of the measures of the corresponding sides of two quadrilaterals are equal, will corresponding angles be congruent?

4. Suppose that ΔABC and $\Delta A'B'C'$ are two similar triangles such that $AB = 3$, $AC = 6$, and $BC = 7$. Find the measures of the lengths of the remaining two sides of $\Delta A'B'C'$ when the measure of one side is as follows:

   (a) $A'B' = 6$.

   (b) $A'C' = 2$.

   (c) $B'C' = 5$.

5. Let ΔABC and ΔTSR be two triangles such that $\angle A \cong \angle T$, $\angle B \cong \angle S$, and $\angle R \cong \angle C$. Will the triangles be similar? Explain.

6. Suppose in the previous problem we knew only that $\angle A \cong \angle R$ and that $\angle B \cong \angle S$. Would it be necessary that the triangles be similar? Explain.

Exercises 9-5

1. Draw a triangle ABC. Let D be the midpoint of side AB and E the midpoint of side AC. Which of the following pairs of ratios are equal? Give reasons for your answers.

   (a) $\frac{AB}{AC}, \frac{AD}{AE}$ (b) $\frac{AB}{AD}, \frac{AC}{AE}$ (c) $\frac{AD}{DE}, \frac{AB}{BC}$ (d) $\frac{AD}{AC}, \frac{AB}{AE}$

2. Would your answers in Problem 1 be different if you had started with a different triangle ABC? Why or why not?

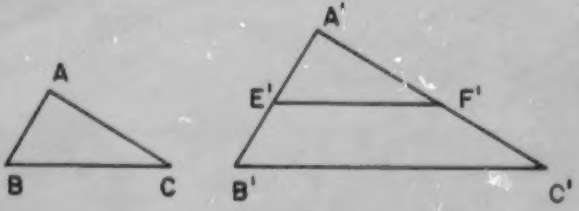
3. Draw a triangle ABC and let $\overline{DE}$ be a line segment parallel to $\overline{BC}$, where D is on $\overline{AB}$ and E is on $\overline{AC}$. Then answer the questions in Problem 1.

4. Suppose ABC is a right triangle and the angle at A has a measure in degrees of 31. What is the measure of the other acute angle?

5. Suppose ABC is a triangle for which $m(\angle A) = 35$ and $m(\angle B) = 47$. Find $m(\angle C)$.

6. Draw a triangle ABC and choose E as some point on the side $\overline{AB}$. Draw a line through E parallel to $\overleftrightarrow{BC}$ and let it intersect $\overline{AC}$ in the point D. Show that triangles ABC and AED are similar and point out which angle in the triangle AED corresponds to $\angle$ ABC in triangle ABC.

7. Let ABC and A'B'C' be two triangles in which angles A and A' are congruent, A'B' = 2(AB), A'C' = 2(AC). Let E' and F' be the midpoints of the sides $\overline{A'B'}$ and $\overline{A'C'}$ of the second triangle. Show that triangle ABC is congruent to triangle A'E'F'. Use this to show that triangle A'B'C' is similar to triangle ABC.



8. Draw triangles ABC and A'B'C' as in Problem 7. Angles A and A' are congruent, but A'B' = 3(AB), A'C' = 3(AC). Show that triangles ABC and A'B'C' are similar.

*9. Suppose ABC and A'B'C' are two triangles and that

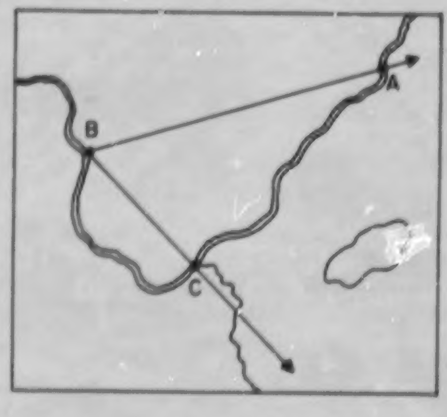
$$\frac{AB}{AC} = \frac{A'B'}{A'C'}, \qquad \frac{AB}{BC} = \frac{A'B'}{B'C'}$$

   (a) If the given equalities hold and A'B' = 3(AB), show that A'C' = 3(AC) and B'C' = 3(BC).

   (b) If the given equalities hold and A'B' = s(AB), show that A'C' = s(AC) and B'C' = s(BC).

9-6. Scale Drawings and Maps

In the previous section you learned to define two triangles as similar if there is a one-to-one correspondence between the vertices such that corresponding angles are congruent and corresponding sides are proportional.



You are familiar with maps. What do you expect from a map? If it is a map of a state, for example, you need to have a one-to-one correspondence between the important cities and the points on the map; that is, each city is to be represented by one particular point on the map as shown at the right. Then, corresponding angles should be congruent; that is, if A, B, and C are three cities, the angle ABC on land should be congruent to the corresponding angle ABC on the map.

Should corresponding distances be equal? This would be too much to expect since then the map would be too large. But if city A is twice as far from city B as city C is, then the corresponding distance on the map, AB, should be twice the correspondin distance BC. In other words, corresponding distances should be proportional in the same way that the sides of two similar triangles are proportional.

Since the earth is a sphere, and there are hills and valleys, no map on a flat sheet of paper can exactly meet the requirements we have set down. In fact, the size of a city is not usually proportional to the size of the dot which represents it on the map. Nor are the widths of a road or a river proportional to the width of the lines which represent them. However, our usual maps approximately satisfy the requirements we gave above.

You have heard of scale drawings. Scale drawings are a kind of map. Such a drawing in a plane must show angles accurately. To make distances the same in the drawing as in the actual object would result in an unmanageable drawing due to size. Hence, we make distances on the drawing smaller than the actual object. But we keep the ratio of the distance on the drawing to the corresponding distance on the object the same. That is, if one length on the object is twice another length, then, on the drawing, the corresponding length will be twice the other. This ratio is called the "scale." Suppose we make a drawing and decide to let one inch on the drawing correspond to one foot on the object. We have chosen

one inch ⟷ one foot

to be the scale, and we must be careful to use this throughout the drawing. Such a scale is usually written "one inch = one foot." If we are careful to make the drawing accurate we can measure distances on the drawing and thus find approximately what the corresponding distances are on the object itself.

The first step in scale drawing is to select a scale so that the drawing will fit on the paper and yet not be too small to use for actual measurement. Suppose you are to make a scale drawing of a football field; that is, the portion marked by lines within which the play takes place. A football field is a rectangle that is 300 feet (that is, 100 yards) long and 160 feet wide. We might try a scale of 0.1 cm to a foot; that is, 0.1 cm on the drawing will correspond to 1 foot on the field. Then on the scale drawing, or map, the length of the field will be represented by a distance of 30 cm. But this is too long for the usual sheet of notebook paper, so we decide to use a "smaller" scale. "Smaller" here means we will use a shorter distance on the scale drawing to correspond to a foot on the field. We select a scale,

$$0.05 \text{ cm} = 1 \text{ foot.}$$

This could also be written as

$$1 \text{ cm} = 20 \text{ feet.}$$

Using this scale, the length of the football field in the drawing will be:

$$(0.05) \cdot (300) = 15.00, \text{ or } 15 \text{ cm}$$

or

$$\frac{300}{20} = 15, \text{ which is also } 15 \text{ cm.}$$

Similarly, we can multiply 160 by 0.05 (or we can divide 160 by 20) to determine that the width of the field will be represented by a line 8 cm long in the drawing.

Class Exercises 9-6

1. Make a scale drawing of a football field. Use the scale 0.05 cm = 1 foot or 1 cm = 20 feet. The length of the field on your drawing should be 15 cm, and the width of the field on your drawing should be 8 cm.
   (a) Draw a line from one corner of the field diagonally across to the opposite corner. Measure the length of this diagonal. What is its length in centimeters?
   (b) Using your scale, determine the measure of the corresponding distance on the football field.
   (c) Verify your answers for (a) and (b) using the Theorem of Pythagoras.

2. Assume you are to make a scale drawing of a football field using the scale $\frac{1}{16}$ inch = 1 foot.
   (a) What would be the length of the field on a scale drawing? (Determine your answer to the nearest tenth of an inch.)
   (b) What would be the width of the field on a scale drawing?

3. What length would a line segment on a scale drawing be to correspond to a measurement of 50 feet for each of the following scales?
   (a) $\frac{1}{2}$ inch = 1 foot.
   (b) 1 mm = 1 foot.
   (c) $\frac{1}{8}$ inch = 1 foot.
   (d) 1 inch = 10 feet.
   (e) $\frac{1}{4}$ inch = 5 feet.
   (f) 0.05 cm = 1 foot.

4. On a scale drawing, the measurement of a line segment is 10 inches. What is the length of the corresponding line for each of the following scales?
   (a) 1 inch = 10 feet.
   (b) $\frac{1}{2}$ inch = 1 foot.
   (c) $\frac{1}{8}$ inch = 1 foot.
   (d) $\frac{1}{4}$ inch = 2 feet.
   (e) 1 inch = 0.5 feet.
   (f) $\frac{1}{2}$ inch = 10 feet.

An illustration in a dictionary may indicate a scale of $\frac{1}{300}$. Similarly, the scale on a map may be $\frac{1}{50,000}$. The first scale means that one unit of measure on the illustration represents 300 units of measure on the actual object. Similarly, the second scale means that one unit of measure on the map represents 50,000 units of measure on the actual object.

A picture of a whale indicates the scale $\frac{1}{300}$. The length of the whale in the picture is about $1\frac{1}{2}$ inches. This means the actual length of the whale is about

$300 \cdot 1\frac{1}{2} = 450$, that is 450 inches or about $37\frac{1}{2}$ feet.

A line segment on a map has a measurement of 5 inches. The scale on the map is $\frac{1}{50,000}$. What is the corresponding measurement on the earth?

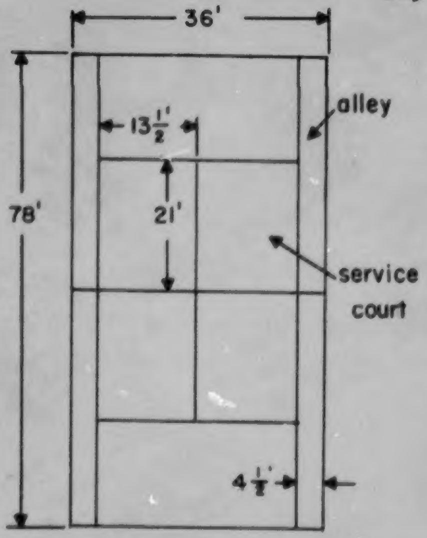
$50,000 \cdot 5 = 250,000$, that is, 250,000 inches.

Dividing 250,000 by 12 we find that the corresponding distance on earth has a measurement of about 20,866.6 feet. Dividing this by 5,280 we determine that this is about 3.9 miles.

Exercises 9-6

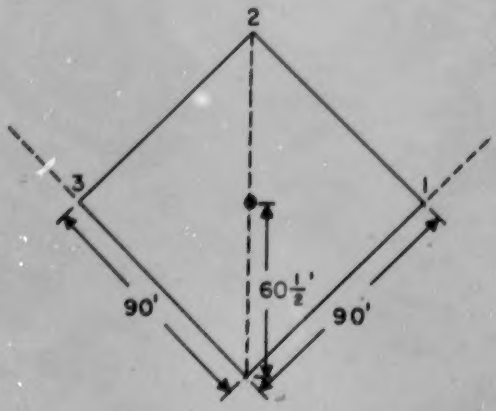
1. It is indicated on a map that $\frac{1}{4}$ inch represents 50 miles. How many miles are represented by $4\frac{1}{4}$ inches?
2. If it is indicated on a map that $\frac{1}{8}$ inch represents 25 miles, how many inches would you use to represent 750 miles?
3. A plot of ground is in the form of a parallelogram. The longer sides measure 92 feet and the shorter sides measure 40 feet. The acute angles have a measurement of 70° and the obtuse angles have a measurement of 110°. Make this scale drawing. Let $\frac{1}{16}$ inch represent 1 foot.
4. City B is 40 miles east of city A and city C is 30 miles north of city B. Using $\frac{1}{8}$ inch to represent one mile draw a map in the form of a scale drawing of these distances. How many miles is it from city A to city C in a direct line?

5. Make a scale drawing of a tennis court. A tennis court is a rectangle having a length of 78 feet and a width of 36 feet. The alleys are $4\frac{1}{2}$ feet wide. The service courts are 21 feet in length and $13\frac{1}{2}$ feet in width. Let $\frac{1}{8}$ inch represent 1 foot.



6. The scale on a map is $\frac{1}{1,200,000}$. To the nearest tenth of a mile, what measurement on earth is represented by a line segment 1 inch long on the map.

7. A regulation baseball diamond is in the shape of a square, the length of whose side is 90 feet. The pitcher's box is on the line through home plate and second base and is $60\frac{1}{2}$ feet from home plate. Make a scale drawing of a baseball diamond locating the pitcher's box as a point. Use measurements from your drawing to answer the following questions.



(a) What is the approximate measurement of the distance from the pitcher's box to second base? (Measure to the vertex of the angle at second base.)

(b) If a person runs directly from first to third base, about how close to the pitcher's box does he come?

(c) If the shortstop stands on the line from second to third base and is halfway between them, about how far is he from home plate?

(d) If a player runs from third base to home plate, what is the measurement of the least distance to the pitcher's box?

8. A ship is seen from two different points, A and B, on the shore. This distance between A and B is 100 feet. If S represents the point where the ship is, the measurements of the following angles are found:

$$m(\angle SAB) = 30^\circ \quad \text{and} \quad m(\angle SBA) = 70^\circ.$$

Make a scale drawing making $AB = 5$ inches.

(a) Why is the triangle you drew similar to the triangle ABS?

(b) By measuring your drawing, find the approximate distances from A to the ship and from B to the ship.

9. Suppose A, B, and C represent three cities the following distances apart:

$$AB = 100 \text{ miles}, \quad AC = 75 \text{ miles}, \quad BC = 60 \text{ miles}.$$

City C is due west of city A, and city B is north of the line AC. Make a scale drawing in which 1 inch corresponds to 20 miles.

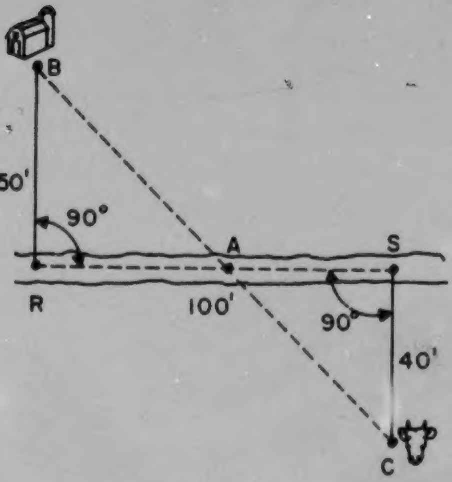
(a) What are the corresponding distances on your scale drawing for AB, AC, and BC?

(b) What is the approximate measurement of the angle between AC and AB, $(\angle BAC)$, on your drawing?

(c) Determine the approximate direction from city A to city B.

10. If in the previous problem, the scale had been chosen so that 1 inch corresponds to 10 miles, what would be your answers for (a)? What would be your answers for (b) and (c)?

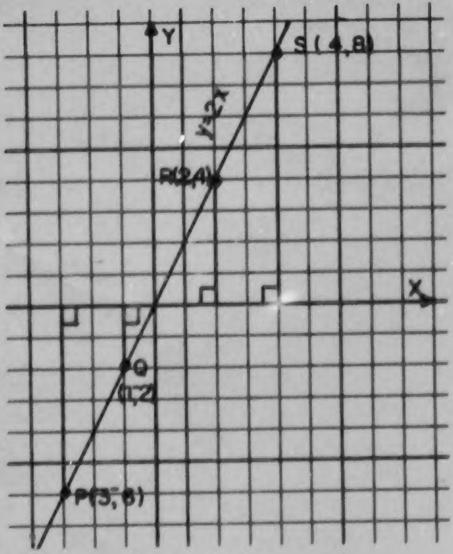
11. A cow and a barn are on opposite sides of a small brook which flows along a straight line. Suppose the cow is 40 feet from the brook and the barn is 50 feet from the brook, both distances being measured perpendicular to the brook. Let 100 feet be the distance between the points on the stream from which these other two distances are measured. Make a scale drawing and use this to find the following distances:



(a) The approximate length of the shortest path from the cow to the barn, B to C.

(b) If this path crosses the stream at A, find the distance from S to A and from A to R.

*12. Devise some method to do Problem 9 without a scale drawing and then verify your answers.



9-7. Kinds of Variation

Earlier in this chapter you learned about similar right triangles. In the drawing at the right the graph of the equation $y = 2x$ is shown. From the points P, Q, R, and S, perpendiculars are dropped to the X-axis forming four similar right triangles. For each of these triangles, the ratio of the y-coordinate to the x-coordinate is equal to $\frac{2}{1}$. Thus, 2 is the slope of the line. There is a relation between the y-coordinate and the x-coordinate for each point on this graph, and this relation is expressed by the equation

$$y = 2x.$$

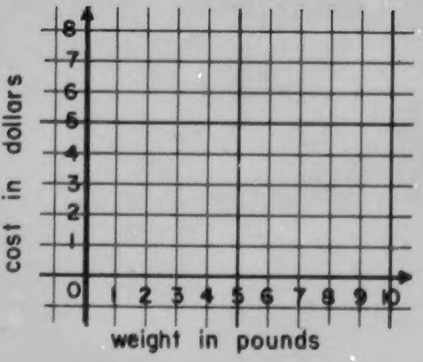
For each ordered pair associated with a point on the graph the y-coordinate is two times the x-coordinate. We say that these two coordinates are related in a particular way. In the world around you there are many situations where two quantities are related. When you buy peanuts, the amount you pay depends on two things, the amount you buy and the price of the peanuts. If you hang a mass on a spring balance, the distance that the spring stretches is related to the strength of the spring and the weight of the mass.

Class Exercises 9-7

1. Suppose peanuts cost $.60 per pound. Make a table showing the cost of various amounts of peanuts as shown below:

Amount in pounds	0	1	2	3	4	5	6	7	8	9	10
Cost in dollars											

Now make a graph showing how the cost is related to the weight. Let w be the weight in pounds (X-axis) and c be the cost in dollars (Y-axis) as shown at the right.

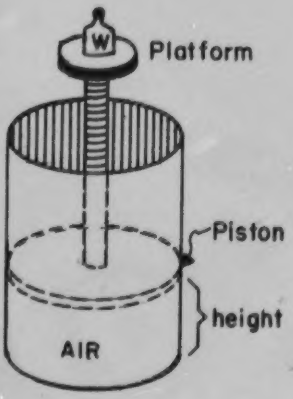


(a) If the number of pounds purchased is increased, what happens to the cost?

(b) What is the slope of the line?

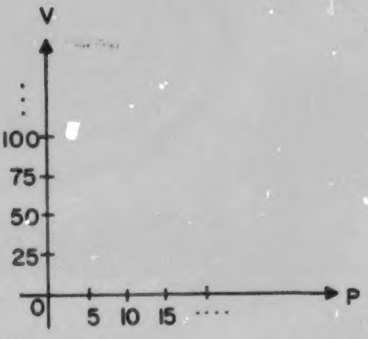
(c) Write an equation expressing the relation between the y-coordinate and the x-coordinate for each point on the line.

2. At the right we have pictured a cylinder filled with air. Pressure can be exerted by placing a weight on the platform P which is connected to a piston in the cylinder. The volume occupied by the air depends on the amount of pressure, which, in turn, depends on the weight at P. Masses of various weights are placed on the platform P of the piston. A pressure gauge is used to measure the pressure p in pounds per square inch of the air in the cylinder. The height of the piston is measured each time and v, the volume, in cubic inches, of air in the cylinder is determined. Here is a table showing the results:



p	15	20	25	30
v	250	187.5	150	125

Make a graph showing the relation between p and v and use the graph in answering the following.



(a) Predict the measure of the volume if $p = 40$.

(b) As the weight increases, what happens to the volume of the air?

(c) If the pressure is 10, predict the measure of the volume.

(d) As the volume increases, what happens to the pressure?

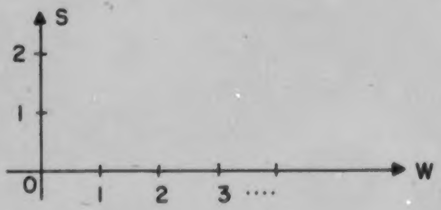
These are but two examples of the many situations where two quantities are related. In these examples you saw that when one of the quantities changes, then the other one also changes in a certain definite way. Mathematicians sometimes say a quantity "varies" when the quantity changes. In the rest of this chapter we shall study some of the simplest properties related to changes. These properties are sometimes called laws of variation.

Exercises 9-7

1. The distances through which a spring is stretched when masses of various weights are hung on it are given in the table. Make a graph showing the relation between the weight w in pounds and the stretch s in inches.

Weight in pounds	0	1	2	3	4
Stretch in inches	0	$\frac{3}{4}$	$1\frac{1}{2}$	$2\frac{1}{4}$	3

(a) As w increases, what happens to s?



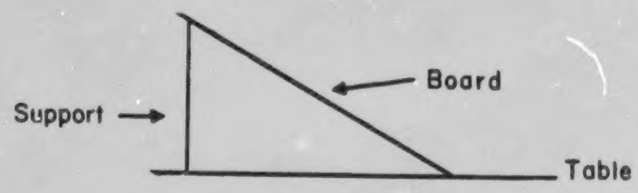
(b) As s increases what happens to w?

(c) Predict the measure of s when the measure of w is 8.

2. A scale in inches is marked off on a board, like this:

0	1	2	3	4	5	6	7	8	9	10	11

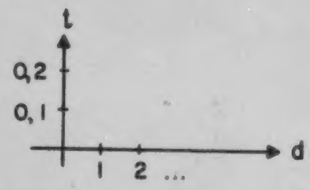
Then it is propped up like this:



Suppose one places a marble at various distances from the bottom and, with a stop-watch, measures how long it takes for the marble to roll to the bottom. Here are the results, with the time measured to the nearest $\frac{1}{10}$ of a second:

d (measure of the distances in inches)	1	2	3	4	5	6	7	8	9	10
t (measure of the time in seconds)	.1	.3	.3	.4	.4	.5	.5	.6	.6	.6

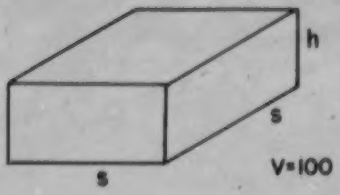
Make a graph showing the relation between d and t.



(a) As the measure of d varies (changes), does the measure of t vary? Explain.

(b) Can you predict about how long it will take the marble to roll 12 inches?

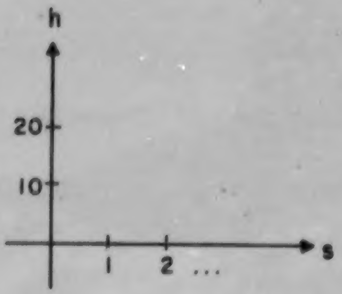
3. Consider a box, with a square base, of volume 100 cubic inches. Make a table showing the relation between the measure s of the length of the side of the base and the measure h of the height.



s	1	2	3	4	5	6	7	8	9	10
h										

(a) Make a graph showing the relation between s and h.

(b) Give a formula expressing h in terms of s.



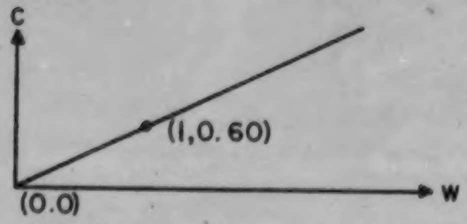
9-8. Direct Variation

In Problem 1 of the last set of exercises, how would you find the cost of 15 pounds of peanuts? Of course, you would say, "One pound of peanuts costs 0.60 dollars. To find the cost of 15 pounds I must _________ (add, subtract, multiply, or divide?) the cost per pound by the number of pounds. The answer is _________ (What is it?)." You can express this process of calculation by means of an equation:

$$c = (0.60)\ \_\_\_\_\_\_\_ \quad \text{(What operation must be performed?)}$$

When you graphed this relation, you obtained a straight line through the origin:

[sec. 9-8]



What is the equation of this line? What is the slope of this line? If you increase the amount you buy by 10 pounds, how much does the cost increase? What is the change in the cost for each unit change in the weight?

We say that c varies directly as w, or that the cost is proportional to the weight. In this relation, the ratio of the measure of the cost to the measure of the weight is always 0.60. Because this number does not change, we call it a constant. We say that 0.60 is the constant of proportionality.

If your father drives along a straight road at the speed of 50 miles per hour, how far does he go in 1 hour? 2 hours? 3.5 hours? t hours? The measure d of the distance traveled is given in terms of the measure t of the time by the formula:

$$d = 50t.$$

We see that the measure of the distance is a constant times the measure of the time. The ratio $\frac{d}{t}$ of the measure of the distance to the measure of the time is a constant. The distance varies directly as the time. We may also say that the distance is proportional to the time. The constant of proportionality is 50.

In the graph of the equation $y = 2x$, the y-coordinate of the points on the graph is a constant times the x-coordinate. In other words, y varies directly as x. We can also say the y-coordinate is proportional to the x-coordinate and the constant of proportionality is 2, the slope of the line.

Class Exercises 9-8

1. According to Hooke's law of elasticity, the amount that a spring stretches is proportional to the weight of the object hung on it. Suppose you know that when a mass having a weight of 2 pounds is hung on the spring, the spring stretches 3 inches. How much stretch would be produced by an object weighing 5 pounds?

   (a) We may express Hooke's law by means of the equation:

   $$s = k \cdot w$$

   where k is some constant of proportionality,

   s is the measure of the stretch of the spring in inches,

   w is the measure of the weight in pounds.

   Replacing s and w with 3 and 2 results in the following equation,

   $$3 = k \cdot 2.$$

   Solve to find the constant of proportionality, k.

   (b) Replace the k in the equation $s = k \cdot w$ with your answer from (a) and solve to find s when $w = 5$.

   (c) Copy and complete the following table using your answer from (a) as the constant of proportionality.

w	0	1	2	3	4	5
s						

   (d) Does the measure of s vary directly with the measure of w?

2. Assume the cost of gasoline is 32 cents per gallon.

   (a) Write an equation in mathematical terms showing the total cost in cents, t, of n gallons of gasoline.

   (b) Write an equation showing the cost as a number of dollars, d, for n gallons of gasoline at 32 cents per gallon.

(c) Does the measure of t vary directly with the measure of n?

(d) Does the measure of n vary directly with the measure of t?

(e) What is the constant of proportionality in this problem?

Exercises 9-8

1. (a) Write a sentence in mathematical terms about the total cost, t cents, of n gallons of gasoline at 33.9 cents per gallon.

   (b) In this statement the cost may also be stated as d dollars. Write the sentence a second way, using d dollars.

2. (a) If your pace is normally about 2 feet, how far will you walk in n steps?

   (b) Use d feet for the total distance and write the formula.

   (c) If n increases, can d decrease at the same time?

3. (a) Write a formula for the number i of inches in f feet.

   (b) As f decreases what happens to i?

4. State the value of the constant, k, in each of the equations you wrote for Problems 1 to 3.

5. Can you write the equation in Problem 2 in the form $\frac{d}{n} = 2$? What restriction does this form place on n?

6. Find k if y varies directly as x, and y is 6 when x is 2.

7. Find k if y varies directly as x, and y is $^-3$ when x is $^-12$.

8. (a) Sometimes it is required to write an equation from a given set of values. From the information in the table, does it appear that a varies directly as b? Why?

(b) What equation appears to relate a and b?

a	b
20	2
50	5
$^-70$	$^-7$
100	10
$^-110$	$^-11$

9. Suppose that d varies directly as t and that when t is 6, d is 240. Write the equation relating d and t.

10. (a) Use the relation $y = \frac{3x}{2}$ to supply the missing values in the following ordered pairs: $(^-4,\ )$; $(^-3,\ )$; $(^-2,\ )$; $(^-1,\ )$; $(0,\ )$; $(2,\ )$; $(5,\ )$.

(b) Plot the points on graph paper.

11. (a) In the relation of Problem 10, when the number x is doubled, is the corresponding number y doubled?

(b) When x is halved, what happens to y?

(c) When the number y is multiplied by 10 what happens to the corresponding number x?

(d) Are your statements true for negative values of x and y?

12. (a) In the equation $y = kx$ what happens to the number x if y is halved?

(b) What happens to y if x is tripled?

13. (a) Give an example of direct variation when the constant of proportionality is a large number.

(b) Give another example where $1 > k > 0$.

9-9. Inverse Variation

Suppose you have 12 quarts of punch for a party, and you want to be perfectly fair to your guests and serve each one exactly the same amount. How does the amount for each guest vary with the number n of guests? For example, if the number of guests is doubled, what is the change in the amount that each one gets? Let q be the number of quarts of punch per guest. Then, the total amount of punch, which is 12 quarts, is equal to:

(the number of quarts per guest) $\cdot$ (number of guests).

The relation between q and n can be expressed by means of the equation,

$$q \cdot n = 12,$$

where the constant of proportionality is 12.

A table of values for the equation $q \cdot n = 12$ is shown below. A table of values for the equation $y = 3 \cdot x$ is shown for comparison. Check the tables and compare the values shown.

$q \cdot n = 12$

q	1	2	3	4	5	6
n	12	6	4	3	$\frac{12}{5}$	2

$y = 3 \cdot x$

x	1	2	3	4	5	6
y	3	6	9	12	15	18

In the table of values for $y = 3 \cdot x$, y increases as x increases. Actually, there is a more definite relationship: if any entry for x is doubled, so is the corresponding entry for y; if the entry for x is tripled, so is the corresponding entry for y. In fact, if an entry for x is multiplied by any number, the corresponding entry for y is multiplied by this same number. This is what we mean by saying "y varies directly as x." This, then, is an example of direct variation.

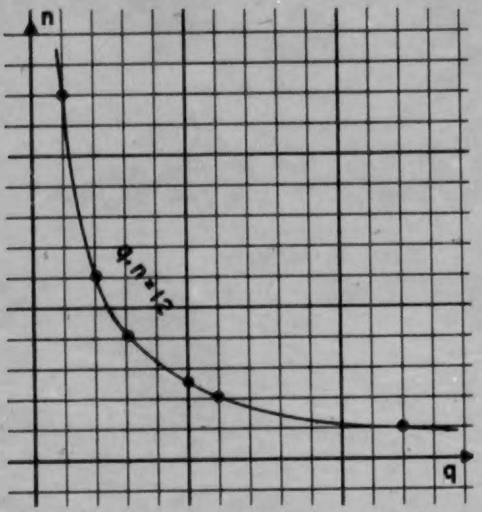
The first table of values is quite different. Here, n decreases as q increases. There is a more definite relationship: if any entry for q is doubled, the corresponding entry for n is divided by 2; if an entry for q is tripled, the corresponding entry for n is divided by 3. In fact, if an entry for q is multiplied by any number, the corresponding entry for n is divided by this same number. We express the relationship $q \cdot n = 12$ by saying:

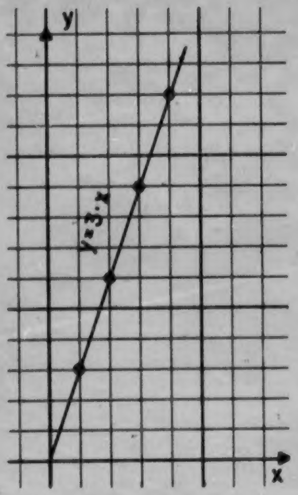
"q varies inversely as n."

Another way of saying the same thing is:

"q is inversely proportional to n."

The graphs of $q \cdot n = 12$ and $y = 3 \cdot x$ are shown below for positive values of q, n, y and x. Notice how what we have said above is shown in the graphs.





Exercises 9-9

1. (a) The table below, as it is now filled in, shows two possible ways in which a distance of 100 miles can be traveled. Copy and complete the table.

Rate (mi. per hr.)	10	20	25	50	60	75	80	100
Time (hours)	10	5						

(b) From part (a), use r for the number of miles per hour and t for the number of hours and write an equation connecting r and t and 100.

(c) When the rate is doubled what is the effect upon the corresponding time value?

(d) When t increases in $rt = 100$ what happens to r?

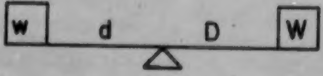
2. (a) Suppose you have 240 square patio stones (flagstones). You can arrange them in rows to form a variety of rectangular floors for a patio. If s represents the number of stones in a row and n represents the number of rows, what are the possibilities? Copy and fill in a table like this one.

Total Number of Stones: 240

Number of stones in a row	10	12	15	16	30	40
Number of rows	24					

(b) Write an equation connecting n, s, and 240. (If you cannot cut any of the stones, what can you say about the kind of numbers n and s must be?)

3. (a) A seesaw will balance if $wd = WD$ when an object weighing w pounds is d feet from the fulcrum and on the other side an object weighing W pounds is D feet from the fulcrum. If $WD = 36$, find D when W is 2, 9, or 18 and find W when D is 1, 6, 12.



(b) What can you say about corresponding values of W as D is doubled if wd remains constant? As numbers substituted for W increase, what can you say about corresponding values of D provided wd remains constant?

4. Write an equation connecting rate of interest r and the number of dollars on deposit p with a fixed interest payment of $200 per year. Discuss how corresponding values of r are affected as different numbers are substituted for p. If the interest rate were doubled how much money would have to be on deposit to give $200 interest per year?

5. Give the constant of proportionality in each of the Problems 1 to 4.

6. State your impression of the difference between direct variation and inverse variation.

7. Find k if y varies inversely as x and if y is 6 when x is 2.

8. Find k if x varies inversely as y and if y is 10 when x is $\frac{1}{2}$.

9. From the information in the table does it appear that a varies inversely as b? Explain your answer.

a	${}^{-}4$	${}^{-}1$	1	3	8	19	41
b	${}^{-}8$	${}^{-}2$	2	6	16	38	82

10. Study the number pairs which follow: $({}^{-}2, 8)$; $({}^{-}1, 2)$; $(0, 0)$; $(1, 2)$; $(2, 8)$; $(3, 18)$; $(4, 32)$.

    (a) Does it appear that y varies directly as x?

    (b) Does it appear that y varies inversely as x?

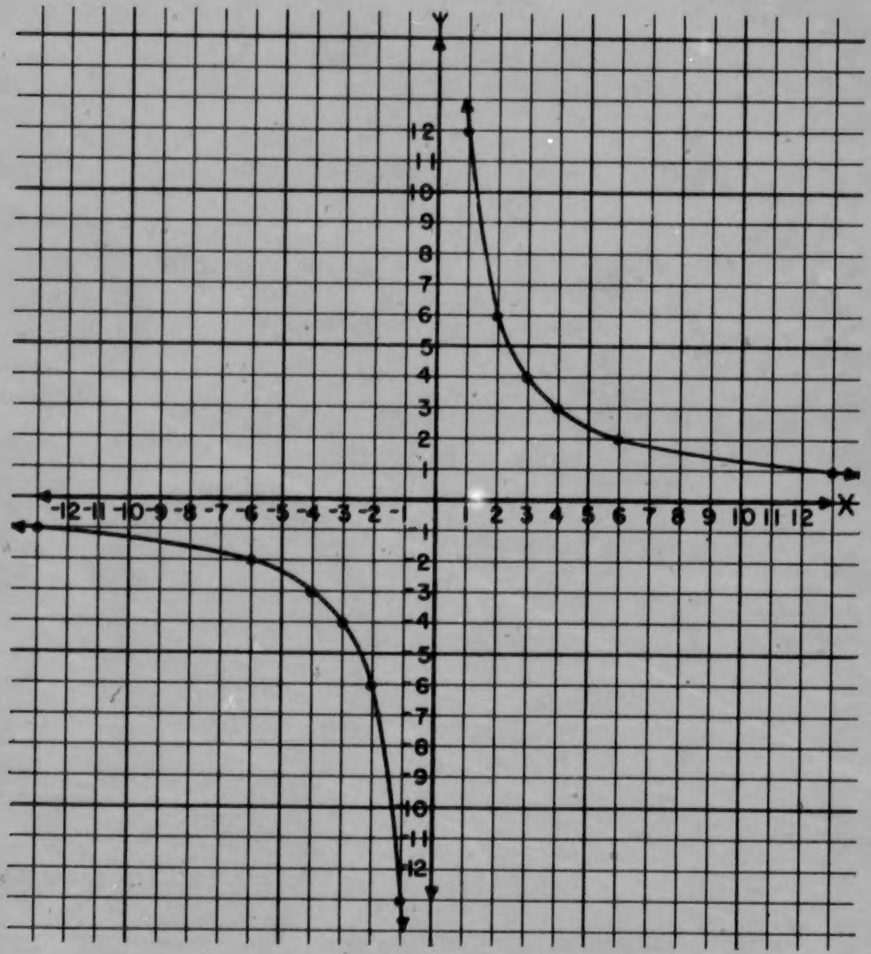
11. (a) Supply the missing values in the table below where $xy = 18$.

x	${}^{-}4$	${}^{-}3$	${}^{-}2$	${}^{-}1$	1	2	3	4	5	6	7	8	9	18
y														

    (b) Is it possible for x or y to be zero in $xy = 18$? Why?

    (c) Plot on graph paper the points whose coordinates you found in (a) and draw the curve. You may wish to find more number pairs to enable you to draw the curve more easily. Does your curve look like the one following (d)?

(d) Some of you may have met this curve in the seventh grade in connection with the lever, or balance. The curve, of which your graph is a portion, is called a hyperbola. It has two branches. A hyperbola shows ________ variation.



12. On graph paper, draw the graph of $xy = 6$.

Notice that in the discussion of direct and inverse variations, the letters x and y may be used interchangeably. In $y = kx$, if k is not zero, we say that x varies directly as y and y varies directly as x. In $xy = k$, k cannot be zero and we say that x varies inversely as y or that y varies inversely as x. In these statements x and y can represent different pairs of numbers while k represents a constant, that is, a fixed number. In the general equation the letter "k" is used rather than a particular numeral, in order to include all possible cases. Since for $x \neq 0$, the equation $xy = k$ may be written $y = k \cdot \frac{1}{x}$ which says that y varies directly as the reciprocal of x.

Occasionally you may see direct variation represented by the statement $\frac{y}{x} = k$. There are times when this form is useful, but from your work with zero you know that $\frac{y}{x} = k$ excludes the possibility of x being zero.

The graphs of $y = kx$ and $xy = k$ include points with negative coordinates. In many problems it doesn't make sense for x or y to be negative. In the problem of serving punch at your party, the number n of guests must be a counting number. In such cases the equation is not a completely correct translation of the relation into mathematical language. The correct translation of your punch-at-the-party problem is the number sentence

(1) "$pn = 12$ and n is a counting number."

The correct translation of Boyle's law is

(2) "$pv = k$ and $p > 0$ and $v > 0$."

When you graph the number sentence (1) you obtain a set of isolated points in the first quadrant. The graph of the relation (2) is the branch of the hyperbola

$$pv = k$$

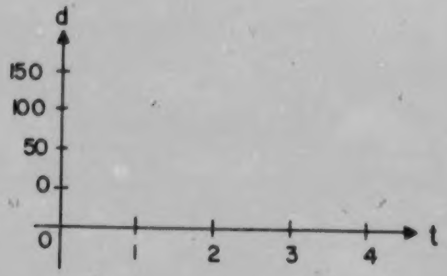
which lies in the first quadrant.

9-10. Other Types of Variation (Optional)

If we make a table of the measure d of the distance in feet through which an object (such as a watermelon) falls from rest in t seconds, we obtain results like this:

t	0	1	2	3	4
d	0	16	64	144	256

Sketch a graph of the relation between d and t. Since the numbers for d are large, you may wish to use a different unit on the d-axis from that used on the t-axis.



You obtain a part of a curve called a parabola.

If the number t is doubled, by how much is the number d multiplied? If t is tripled, by how much is d multiplied? As you see, d varies directly as t^2. We can express the relation by means of the equation

$$d = kt^2,$$

where k is constant. What is this constant? How far does the object fall in 10 seconds?

Galileo discovered this law many years ago. He started with experiments like that with an inclined plane (see Problem 2 of 9-7). From the table we gave, it would be hard to guess the law. More accurate measurements would form the basis for a correct guess.

Exercises 9-10a

1. (a) What is the area of each face of a cube whose sides have length 2 inches? How many faces are there? What is the total surface area, the total area of all faces?

   (b) Make a table showing the relation between the lengths of each side of a cube and the surface area.

s	1	2	3	4	5	6
A						

2. Let S be the measure of the area in square centimeters of a square with edges e centimeters long.

   (a) Find an equation connecting S and e.

   (b) Tell how S varies with e.

   (c) Plot the graph of the equation you found in (a). Use values of e from 0 to 15 and choose a convenient scale for the values of S.

   (d) From the graph you drew in (c), find:

      (1) The area of a square with edges 3 cm. long.

      (2) The length of the edges of a square of area 64 square centimeters.

      (3) The area of a square with edges 5.5 cm. long.

      (4) The length of the edges of a square of area 40 sq. cm.

   (e) From the equation you found in (a), find:

      (1) The area of a square with edges 3 cm. long.

      (2) The area of a square with edges 5.5 cm. long.

3. For each of the observations in Problem 2 of Exercises 9-7 calculate the ratio $\frac{d}{t^2}$. Find the average of these ratios. Make a graph of the equation

$$d = kt^2$$

Using for k the average of the ratios $\frac{d}{t^2}$ which you have just calculated, mark off the points representing the observations. Does the "theoretical curve" fit the experimental data fairly well? Does it seem that $\frac{d}{t^2}$ is nearly constant according to the data?

4. If E is proportional to the square of v and E is 64 when v is 4, find:
   (a) an equation connecting E and v.
   (b) the value of E when $v = 6$.
   (c) the value of v when $E = 16$.

5. Suppose grass seed costs 70 cents per pound, and one pound will sow an area of 280 sq. ft.
   (a) How many pounds of seed will be needed to sow a square plot 10 ft. on a side?
   (b) How much will it cost to buy seed to sow a square plot 10 feet on a side?
   (c) If C cents is the cost of the seed to sow a square plot s feet on a side, find an equation connecting C and s.
   (d) How much will it cost for seed to sow a square plot 65 feet on a side?
   (e) If $15.00 is available for seed, can enough be bought to sow a square plot 75 feet on a side?

6. A ball is dropped from the top of a tower. The distance, d feet, which it has fallen varies as the square of the time, t sec., that has passed since it was dropped.
   (a) From the information above, what equation can you write connecting d and t?
   (b) Find how far the ball falls in the first 3 seconds.

(c) If you are also told that the ball falls 144 feet in the first 3 seconds, write an equation connecting d and t.

(d) Using the equation you wrote in (c), can you find how far the ball falls in the first 5 seconds?

Newton's law of gravitation says that the force with which two objects attract each other varies inversely as the square of the distance between them:

$$F = \frac{k}{d^2},$$

where k is a constant.

The problems in this chapter give you some idea of the many different ways in which two varying quantities may be related to each other. In most of the cases we have discussed, the relation can be expressed by an equation of the form

$$y = kx^n$$

or

$$y = \frac{k}{x^n}$$

where n is a counting number and k is a certain constant. As we have indicated many of the laws of nature are of this type.

Exercises 9-10b

Use the following notation: e cm. is the length of one edge of a cube; P cm. is the perimeter of one face of the cube; S sq. cm. is the total area of all faces of the cube; V cu. cm. is the volume of the cube.

1. Find an equation connecting P and e; S and e; V and e.

2. Complete the following statements:

   (a) P varies _____ _____ _____.

   (b) S varies _____ _____ _____.

   How would you describe the way V varies with e?

3. On one set of axes, plot the graphs of the three equations you found in (a). Use the values 0, 1, 2, 3, 4 for e.

4. From the graphs you drew in Problem 3, find P, S, and V when e is $2\frac{1}{2}$. Check by using the equations you found in Problem 1.

5. Use the graphs you drew in Problem 3 to estimate which of P, S, and V will be greatest and which will be smallest when e is 10. Use the equations you found in Problem 1 to test your guess.

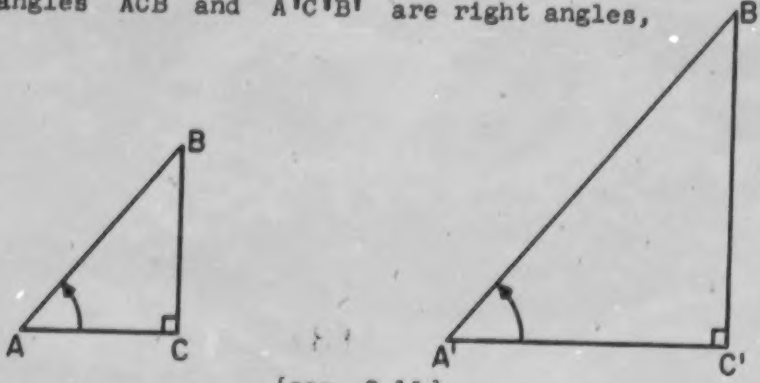
9-11. Summary and Review

Our work on right triangles in this chapter was based on the property: If a pair of corresponding acute angles in two triangles are equal in measurement, then the ratios of the measures of the lengths of corresponding sides are equal.

In mathematical language:

If angles BAC and B'A'C' are equal in measurement

and angles ACB and A'C'B' are right angles,



then in the triangles ABC and A'B'C' we have

$$\frac{AC}{A'C'} = \frac{BC}{B'C'} = \frac{AB}{A'B'}.$$

The following pairs of ratios are also equal:

$$\frac{AC}{AB} = \frac{A'C'}{A'B'}, \quad \frac{BC}{AC} = \frac{B'C'}{A'C'}, \quad \frac{BC}{AB} = \frac{B'C'}{A'B'}.$$

Two triangles ABC and DEF, whether they are right triangles or not, are said to be similar if there is a one-to-one correspondence,

$$A \longleftrightarrow D, \quad B \longleftrightarrow E, \quad C \longleftrightarrow F$$

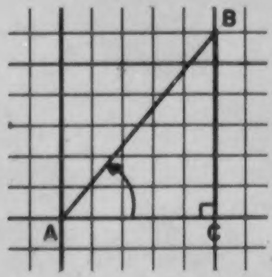
between the vertices such that

(1) corresponding angles are congruent, and

(2) corresponding sides are proportional.

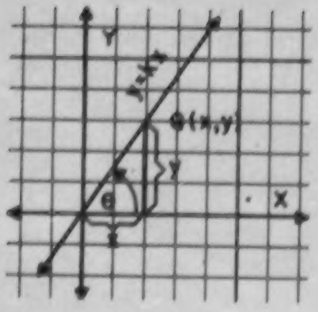
It is stated, though not proved, that either of these conditions implies the other.

In a right triangle the following trigonometric ratios are important



$$\sin \angle CAB = \frac{CB}{AB} \qquad \tan \angle CAB = \frac{CB}{AC} \qquad \cos \angle CAB = \frac{AC}{AB}$$

The equation of a line through the origin has the form $y = kx$, where k is a constant. The number



k is called the slope of the line. If θ (read "theta") is the measurement of the angle which the line makes with the positive ray of the X-axis, then

$$k = \tan \theta$$

If Q (x, y) is any point on the line other than the origin, then

$$\frac{y}{x} = k.$$

As a point Q moves along the line, then

$$\frac{\text{Change in } y}{\text{Change in } x} = k = \text{slope}.$$

Direct application of the idea of similar figures is made to scale drawings and maps.

This relation between two quantities is one of the types of variation considered in this chapter. The three kinds of variation considered in this chapter are direct variation, inverse variation, and direct variation as the square.

(1) Direct variation: $y = kx$

  (a) If x and y are related by the equation $y = kx$, where k is a constant not zero, we say that y varies directly as x. We sometimes omit the word "directly."

  (b) The number k is called the constant of proportionality.

(c) When k is positive, as x increases y must increase and as x decreases y must decrease.

(2) Inverse variation: $xy = k$

(a) If x and y are related by the equation $xy = k$, where k is a constant (not zero), we say that y varies inversely as x.

(b) The number k is the constant of proportionality between y and the reciprocal of x as shown in the form $y = k \cdot \frac{1}{x}$.

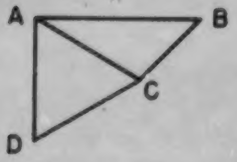
(c) When k is positive, as x increases y must decrease, and as x decreases y must increase.

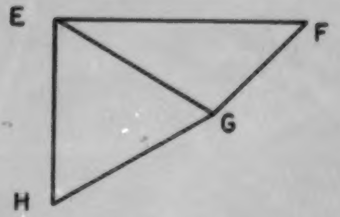
(d) The graph of $xy = k$ is not a straight line, but a curve with two branches. The graph does not go through the origin, and there is no point on the graph for $x = 0$ or for $y = 0$.

The exercises below review the different types of variation discussed in this chapter.

Exercises 9-11

1. If y varies directly as x, and if y is 16 when x is 2, find y when x is 5.
2. If y varies inversely as x, and if y is 16 when x is 2, find y when x is 5.
3. If y varies directly as the square of x, and if y is 16 when x is 2, find y when x is 5.
4. The areas enclosed by two similar polygons are proportional to the squares of any two corresponding diagonals. The polygons on the next page are similar and point C is 2 centimeters from point A; point G is 3 centimeters from point E. Find the ratio of the measures of the regions enclosed by the polygons.





5. The distance, d inches, a spring is stretched varies directly as the pull, P pounds, which is applied to the spring.

   (a) If a pull of 10 lbs. stretches a certain spring 5 inches, what pull is required to stretch it 14 inches?

   (b) For the spring in (a), how far will it be stretched by a pull of 14 lbs.?

6. The pressure, p lbs. per sq. in., exerted by a certain amount of hydrogen gas varies inversely as the volume, v cu. in., of the container in which it is kept. If the pressure is 7 lbs. per sq. in. when the gas is in a gallon jug, what would be the pressure if the gas were enclosed in a half-pint jar?

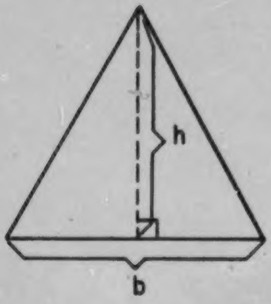
7. Show that, in $x + y = k$, y does not vary inversely as x.

8. If A is 24 in $A = \ell w$, what kind of variation is indicated between ℓ and w?

9. What kind of variation is represented by $C = \pi d$? What is the constant of proportionality?

10. Suppose $V = \pi r^2 h$, and suppose the number r is multiplied by 5 while h is unchanged. What happens to V? What is the constant in this case?

11. Make a table showing the relation between the measure h of the length of the altitude and the measure b of the length of the base in an equilateral triangle. Calculate your results correct to 1 decimal place.



b	2	4	6	8	10
h					

This table can be completed by measurement. Construct equilateral triangles of base 2 units, 4 units, etc. Your results will be more accurate if your triangles are not too small. Make a graph showing the relation between b and h. What simple geometric figure is formed by the graph?

12. Give a formula expressing h in terms of b in Problem 5. You may wish to use $\tan 60^\circ = h \div \frac{b}{2}$.

13. Make a table showing the relation between the measure b of the length of the base and the measure A of the area of an equilateral triangle. Use the values of b which you used in your table for Problem 5. Make a graph showing the relation between b and A. Give a formula for A in terms of b.

Chapter 10

NON-METRIC GEOMETRY

10-1. Introduction

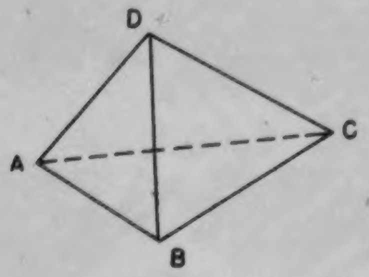
Our earlier work in geometry has been concerned largely with geometric figures in the plane, although a few solids like cubes and cylinders have been considered. The world we see around us is very definitely a 3-dimensional one. Your desk and chair are rather complicated solids. Buildings illustrate numerous curves, surfaces, and solids in 3-dimensional space. To describe the flight of a plane, or of a rocket, or of an artillery shell requires a rather precise knowledge of geometric space.

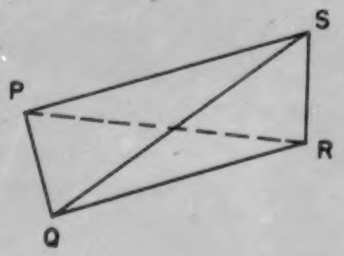
A common automobile gear illustrates a complicated 3-dimensional object that at present we are unable to describe in mathematical terms. The sky, with its planets, stars, satellites, and jets; the earth with its various humps, hollows, and canyons (all placed on top of a more or less "spherical" shape); our roads and superhighways; your own house, school, church, and museum--these all suggest a myriad of geometric figures, forms, and solids which the geometer seeks to describe.

In this chapter, and in Chapter 11 and 12 which follow, we begin some further work in the study of 3-dimensional geometry--the geometry of the space in which we live. Of course, we cannot immediately develop satisfactory methods of describing all the things we use or see or read about. You will learn ways to think about geometric properties in space in terms of simple components and in terms of their similarities to simple figures in the plane. You will learn how to relate space properties to those properties we have learned about points, lines, and circles. You will learn how to investigate and discover some of these properties on your own. In the process you will get a better idea about how the mathematician works to break down complicated geometric figures into simpler components that are easily studied. This type of approach is a valuable one in many respects--valuable alike to the future mathematician, engineer, housewife, businessman, or carpenter.

10-2. Tetrahedrons

A geometric figure of a certain type is called a tetrahedron. A tetrahedron has four vertices which are points in space. The drawings below represent tetrahedrons. Another form of the word "tetrahedrons" is "tetrahedra." "Tetra" is the Greek word for four.



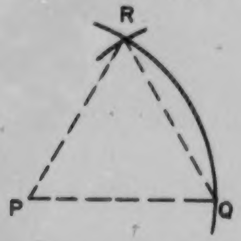


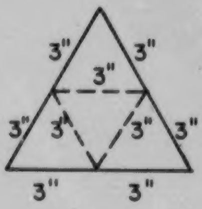
The points A, B, C, and D are the vertices of the tetrahedron on the left. The points P, Q, R, and S are the vertices of the one on the right. The four vertices of a tetrahedron are not in the same plane. The word "tetrahedron" refers either to the surface of the figure or to the "solid" figure--that is, the figure including its interior. From some points of view the distinction is unimportant. Usually, we shall use the term "solid tetrahedron" when we mean the surface together with the interior. We can name a tetrahedron by naming its vertices. Usually we shall put parentheses around the letters like (ABCD) or (PQRS) in naming solid tetrahedrons. The vertices may be named in any order.

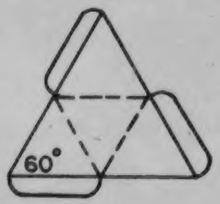
The segments $\overline{AB}$, $\overline{BC}$, $\overline{AC}$, $\overline{AD}$, $\overline{BD}$, and $\overline{CD}$, are called edges of the tetrahedron (ABCD). We sometimes will use the notation (AB) or (BA) to mean the edge $\overline{AB}$. What are the edges of the tetrahedron (PQRS)?

Any three vertices of a tetrahedron are the vertices of a triangle and lie in a plane. A triangle has an interior in the plane in which its vertices lie (and in which it lies). Let us use (ABC) to mean the triangle ABC together with its interior. In other words, (ABC) is the union of Δ ABC and its interior. The sets (ABC), (ABD), (ACD), and (BCD) are called the faces of the tetrahedron (ABCD). What are the faces of the tetrahedron (PQRS)?

We introduce length or measurement here, and occasionally elsewhere, for convenience in making some uniform models and because of greater ease in visualizing the solids. This chapter deals fundamentally with non-metric or "no-measurement" geometry. You will be asked to make some models of tetrahedrons in the exercises. The easiest type of tetrahedron of which to make a model is the so-called regular tetrahedron. Its edges are all the same length. On a piece of cardboard or stiff paper construct an equilateral triangle of side 6". You can do this with a ruler and compass or with a ruler and protractor.





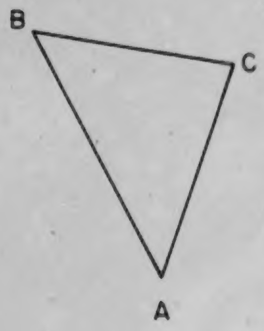


Can you see how the drawing on the left above suggests the construction with ruler and compass? The arc through Q and R has center at P. The other arc through R has the same radius, but its center is at Q. The segments $\overline{PQ}$, $\overline{PR}$, and $\overline{QR}$ have the same length and are congruent.

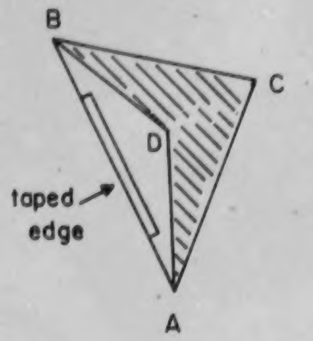
Now mark the three points that are halfway between the pairs of vertices. Cut out the large triangular region. Carefully make three folds or creases along the segments joining the "halfway" points. You may use a ruler or other straightedge to help you make these folds. Your original triangular region now looks like four smaller triangular regions. Bring the original three vertices together above the center of the middle triangle. Fasten the loose edges together with tape or paper and paste. This is easier if you add flaps as in the third figure. You now have a model of a regular tetrahedron.

How do we make a model of a tetrahedron which is not a regular one? Cut any triangular region out of cardboard or heavy paper. Use this as the base of your model as shown below. Label its vertices A, B, and C. Cut out another triangle with one of its edges the same length as $\overline{AB}$. Fasten these two triangles together with tape along edges of equal length; for instance, use edge (AB). Two of the vertices of the second triangle are now considered labeled A and B. Label the other vertex of the second triangle D. Cut out a third triangular region with one edge the length of $\overline{AD}$ and another the length of $\overline{AC}$. If the angle between these edges is too large or too small, the model is more difficult to put together. Now fasten the edges of the third triangle to $\overline{AD}$ and $\overline{AC}$ with tape so that the three triangles fit together in space. The model you have constructed so far will look something like a pyramid-shaped drinking cup if you hold the vertex A at the bottom, as in the drawing below. Finally, cut out a triangular region which will fit the top and fasten it to the top. You now have a model of a tetrahedron.

(1)



(2)



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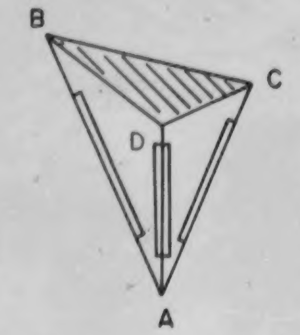


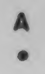
Exercises 10-2

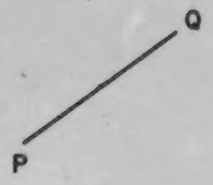
1. Make two cardboard or heavy paper models of a regular tetrahedron. Make your models so that each edge is 3" long.
2. Make a model of a tetrahedron that is not regular.
3. In making the third face of a non-regular tetrahedron, what difficulties would you encounter if you made the angle DAC too large or too small?

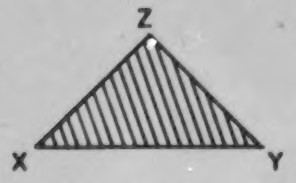
10-3. Simplexes

A single point is probably the simplest set of points you can think of. A set consisting of two points is probably the next most simple set of points. Any two different points in space are on exactly one line, and are the endpoints of exactly one segment (which is a subset of the line). A segment has length but does not have width or thickness, so it does not have area. We speak of a segment or a line as being 1-dimensional. Either could be considered as the simplest 1-dimensional object in space. In this chapter we will think about the segment, not the line.

A set consisting of three points is the next most simple set of points in space. If all three points are on the same line, we get only part of a line. This is the same object that we got with just two points. Let us agree, therefore, that our three points are not to be on the same line. Thus, there is exactly one plane containing the three points and there is exactly one triangle with the three points as vertices. There is also exactly one triangular region which, together with the triangle that bounds it, has the three points as vertices. This mathematical object, the triangle, together with its interior, is what we will think about. It has area and it is 2-dimensional. It can be considered as the simplest 2-dimensional object in space.







The next most simple set of points in space would be a set of four points. If the four points were all in one plane then the figure determined by the four points would also be in one plane. We want to require that four points are not all in any one plane. This requirement also guarantees that no three can be on a line. If any three were on a line, then there would be a plane containing that line and the fourth point, and the four points would be in the same plane. We have four points in space, then, not all in the same plane. This suggests a tetrahedron. The four points in space are the vertices of exactly one solid tetrahedron. A solid tetrahedron has volume, and it is 3-dimensional. It may be considered as the simplest 3-dimensional object in space.

Here we have four sets, each of which may be thought of as the simplest of its kind. Among them are remarkable similarities; they should have names sounding alike and reminding us of their basic properties. We call each of these a simplex. We tell them apart by labeling each with its natural dimension. Thus, a set consisting of a single point is called a 0-simplex. A segment is called a 1-simplex. A triangle, together with its interior, is called a 2-simplex. A solid tetrahedron (which includes its interior) is called a 3-simplex.

Let us make a table to help us keep these ideas in order.

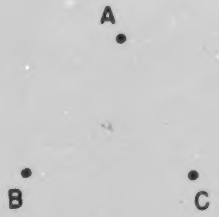
Number of points	Simplest object determined	To be called
1	point	0-simplex
2	segment	1-simplex
3	triangle together with its interior	2-simplex
4	solid tetrahedron	3-simplex

There is another way to think about the dimensions of these sets—the notation of betweenness, or of a point being between two other points.

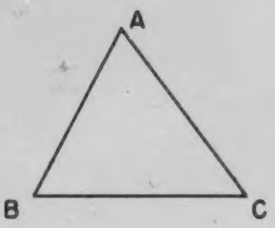
Let us start with two points. Consider these two points and all points between them. The set so formed is a segment. Now take the segment together with all points that are between any two points of the segment. We get just the same segment. No new points were obtained by "taking points between" again. The process of "taking points between" was used just once. We get a 1-dimensional set, a 1-simplex.

Class Exercise 10-3

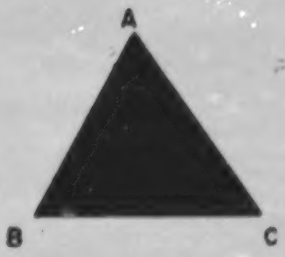
1. Mark three points A, B, and C, not all on the same line, about the same distance apart as in the figure at the right.



(a) Draw line segments including all points between A and B, between B and C, and between A and C. What does the figure represent?



(b) Shade or color all points between any two points of the set determined in (a). What does this figure represent?



(c) If the process in (b) is applied again will you get any new points? How many times did you use the process? The figure you have obtained represents a 2-dimensional set, a 2-simplex.

2. Think of four points A, B, C, and D, not all on the same plane. It will be better in this problem to use your model of a non-regular tetrahedron than to try to draw a figure.

   (a) Shade or color (or draw a segment) which will include all points between A and B. Also, shade in (or draw the segment) containing all points between B and C, between A and C, and so on, until all points between any two vertices are included in your drawing. Describe the set of points you have drawn.

   (b) Shade or color all points between any two points of the set determined in (a). Describe the set of points which are shaded or colored.

   (c) Think of the union of the set of points which are shaded or colored and all points between any two of the shaded or colored points. Describe this new set of points as a union of two sets.

   (d) If the process is repeated again, will you get any new points? How many times did you use the process? The set of points obtained in (c) is a three dimensional set, a 3-simplex.

3. Let us consider just one point.

   (a) If you start with just one point and apply the process used in Problem 1 and 2, what set will you obtain?

(b) How many times will you need to apply the process of "taking all points between any two points"? The set of points obtained is a 0-dimensional set, a 0-simplex.

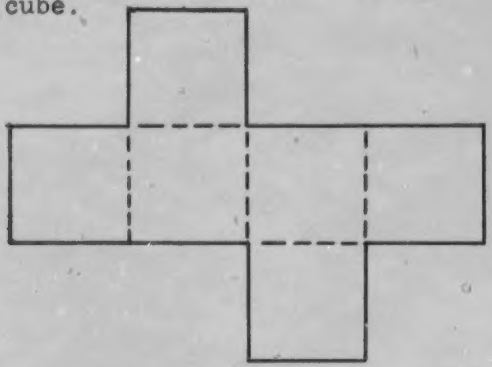
Finally, let us consider a 3-simplex again. Look at one of your models of tetrahedrons. It has four faces, and each face is a 2-simplex. It has six edges and each edge is a 1-simplex, it has four vertices and each vertex is a 0-simplex.

Exercises 10-3

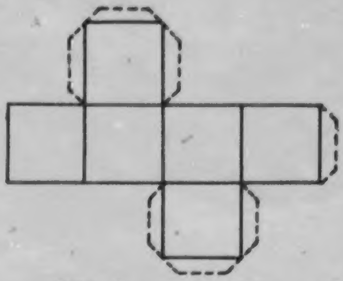
1. (a) A 2-simplex has how many 1-simplexes as edges?
   (b) It has how many 0-simplexes as vertices?
2. A 1-simplex has how many 0-simplexes as vertices?
3. Draw a figure showing how two 1-simplexes can have an intersection which is exactly an endpoint of each.
4. Draw a figure showing how two 2-simplexes can have an intersection which is exactly one vertex of each.
5. Draw a figure showing how two 2-simplexes can have an intersection which is exactly one 1-simplex of each.
6. Using models, show how two 3-simplexes can have an intersection that is exactly one vertex of each.
7. Using models, show how two 3-simplexes can have an intersection that is exactly one edge of each.
8. Using models, show how two 3-simplexes can have an intersection which is exactly one 2-simplex of each.

10-4. Models of Cubes

Most of you already know that if you want to make an ordinary box you need six rectangular faces for it. The faces have to fit and have to be put together correctly. There is a rather easy way to make a model of a cube.



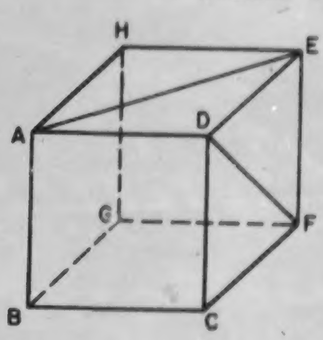
Draw six squares on heavy paper or cardboard as in the drawing above. Cut around the boundary of your figure and fold (or crease) along the dotted lines. Use cellulose tape or paste to fasten it together. If you use paste, it will be necessary to have flaps as indicated in the drawing below.

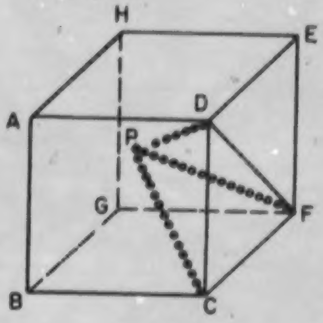


In the exercises you will be asked to make several models of a cube.

Can the surface of a cube be regarded as the union of 2-simplexes (that is, of triangles together with their interiors)? Can a solid cube be regarded as the union of 3-simplexes (that is, of solid tetrahedrons)? The answer to both of these questions is "yes." We shall explain one way of thinking about these questions.

Each face of a cube can be considered to be the union of two 2-simplexes. The drawing on the left below shows a cube with two of its faces subdivided into two 2-simplexes each. The face ADEH appears as the union of (ADE) and (AEH), for example. The other face, which is indicated as subdivided, is CDEF. It appears as the union of (CDF) and (DFE). The other faces have not been subdivided, but we can think of each of them as the union of two 2-simplexes. Thus the surface of the cube can be thought of as the union of twelve 2-simplexes.





With the surface regarded as the union of 2-simplexes we may regard the solid cube as the union of 3-simplexes (solid tetrahedrons) as follows. Let P be any point in the interior of the cube. For any 2-simplex on the surface, (CDF), for example, (PCDF) is a 3-simplex. In the figure on the right above, P is indicated as inside the cube. The 1-simplexes, (PC), (PD), and (PF), are also inside the cube. Thus with twelve 2-simplexes on the surface, we would have twelve 3-simplexes whose union would be the cube. The solid cube is the union of 3-simplexes in this "nice" way.

Exercises 10-4

1. With cardboard or heavy paper, make two models of cubes. Make them with each edge 2" long.

2. On one of your models, without adding any other vertices, draw segments to express the surface of the cube as a union of 2-simplexes. Label all the vertices on the model A, B, C, D, E, F, G, and H. Think of a point P in the interior of the

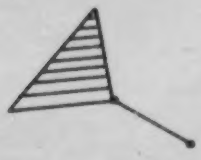
cube. Using this point and the vertices of the 2-simplexes on the surface, list the twelve 3-simplexes whose union is the solid cube.

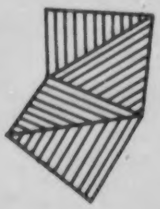
3. On the same cube as in Problem 2, mark a point in the center of each face. (Each should be on one of the segments you drew in Problem 2.) Draw segments to indicate the surface of the cube as the union of 2-simplexes, using as vertices the vertices of the cube and these six new points you have marked. The surface is now expressed as the union of how many 2-simplexes?

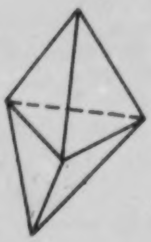
4. Think about a geometric figure formed by putting a square-based pyramid on each face of a cube with the base of the square congruent to the face of the cube. This is one example of a polyhedron. How many triangular faces has the surface of this polyhedron? Can you set up a one-to-one correspondence between this polyhedron, vertex for vertex, edge for edge, and 2-simplex for 2-simplex, and the surface of a cube subdivided into 2-simplexes as in Problem 3?

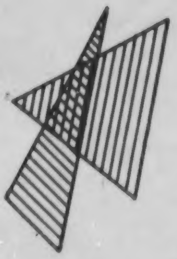
10-5. Polyhedrons

A *polyhedron* is the union of a finite number of simplexes. It could be just one simplex, or perhaps the union of seven simplexes, or perhaps of 7,000,000 simplexes. What we are saying is that a polyhedron is the union of some particular number of simplexes. In the previous section, we observed that a solid cube could be considered as the union of twelve 3-simplexes. The figures below represent the union of simplexes.







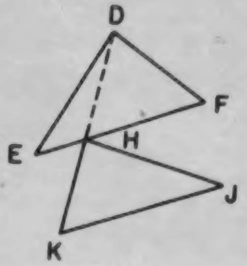


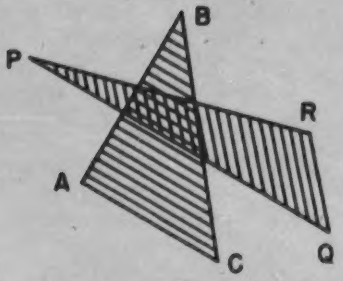
The figure on the left represents a union of a 1-simplex and a 2-simplex which does not contain the 1-simplex. It is therefore of mixed dimension. In what follows, we shall not be concerned with polyhedrons of mixed dimension. We assume that a polyhedron is the union of simplexes of the same dimension. We shall speak of a 3-dimensional polyhedron as one which is the union of 3-simplexes. A 2-dimensional polyhedron is one which is the union of 2-simplexes. A 1-dimensional polyhedron is one which is the union of 1-simplexes. Any finite set of points could be thought of as a 0-dimensional polyhedron, **but we won't be dealing with such here.** A polyhedron does not necessarily consist of one connected part, although most of our examples will be of this type.

The figure on the right above represents a polyhedron which seems to be the union of two 2-simplexes (triangular regions) but they do not intersect nicely. We prefer to think of a polyhedron as the union of simplexes which intersect nicely as in the middle two figures. The third figure shows two 3-simplexes with a 2-simplex as their intersection. Just what do we mean by simplexes intersecting nicely? There is an easy explanation for it.

If two simplexes of the same dimension intersect nicely, then the intersection must be a face, or an edge, or a vertex of each.

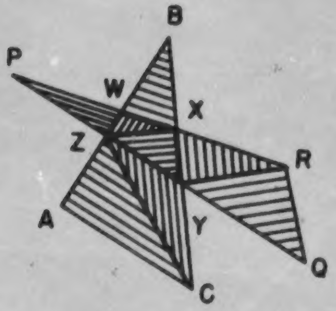
Let us look more closely at the union of simplexes which do not intersect nicely. In the figure on the right the 2-simplexes (DEF) and (HJK) have just the point H in common. They do not intersect nicely. While H is a vertex of (HJK), it is not of (DEF). However, the polyhedron which is the union of these two 2-simplexes is also the union of three 2-simplexes which do intersect nicely, (DEH), (DHF), and (HJK).





The figure on the left represents the union of the 2-simplexes (ABC) and (PQR). They do not intersect nicely. Their intersection seems to be a quadrilateral together with its interior.

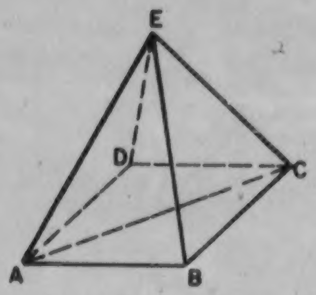
On the right we have indicated how the same set of points (the same polyhedron) can be considered to be a finite union of 2-simplexes which do intersect nicely. The polyhedron is the union of the eight 2-simplexes, (ACZ), (CZY), (PZW), (XYZ), (WXZ), (BWX), (XYR), and (YQR).

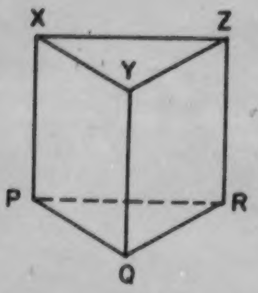


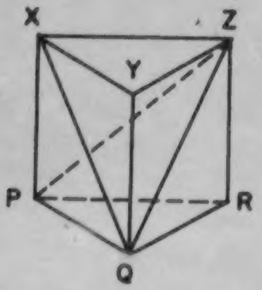
These examples suggest a fact about polyhedrons. If a polyhedron is the union of simplexes which intersect any way at all, then the same set of points (the same polyhedron) is also the union of simplexes which intersect nicely. Except for the exercises at the end of this section, we shall always deal with unions of simplexes which intersect nicely. We will regard a polyhedron as having associated with it a particular set of simplexes which intersect nicely and whose union it is. When we use the word "polyhedron," we understand the simplexes to be there.

Is a solid cube a polyhedron, that is, is it a union of 3-simplexes? We have already seen that it is. Is a solid prism a polyhedron? Is a solid square-based pyramid? The answer to all of these questions is yes. In fact, any solid object each of whose faces is flat (that is, whose surfaces does not contain any curved portion) is a 3-dimensional polyhedron. It can be expressed as the union of 3-simplexes.

As examples let us look at a solid pyramid and a prism with a triangular base.







In the figure on the left the solid pyramid is the union of the two 3-simplexes (ABCE) and (ACDE). The figure in the middle represents a solid prism with a triangular base. The prism has three rectangular faces. Its bases are (PQR) and (XYZ). Here we see how it may be expressed as the union of eight 3-simplexes. We use the same device that we used for the solid cube. First we think about the surface as the union of 2-simplexes. We already have the bases as 2-simplexes. Then we think of each rectangular face as the union of two 2-simplexes. In the figure on the right above, the face YZRQ is indicated as the union of (YZQ) and (QRZ), for instance. Now think about a point F in the interior of the prism. The 3-simplex (FQRZ) is one of eight 3-simplexes each with F as a vertex and whose union is the solid prism. In the exercises you will be asked to name the other seven.

Finally, how do we express a solid prism with a non-triangular base as a 3-dimensional polyhedron—that is, as a union of 3-simplexes with nice intersections? We use a little trick. We first express the base as a union of 2-simplexes and therefore the solid prism as a union of triangular solid prisms. And we can then express each triangular solid prism as the union of eight 3-simplexes. We can do this in such a way that all the simplexes intersect nicely. It may help you to understand this solid if you think of a prism as a solid with flat faces such that two faces called the bases, are congruent and in parallel planes.

There is a moral to our story here. To do a harder-looking problem, we first try to break it up into a lot of easy problems, each of which we already know how to solve.

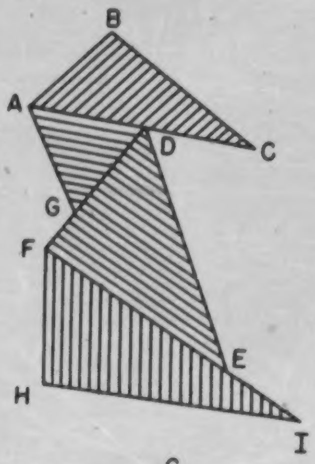
Exercises 10-5

1. Draw two 2-simplexes whose intersection is one point and
   (a) the point is a vertex of each.
   (b) the point is a vertex of one but not of the other.

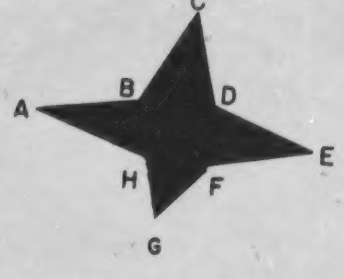
2. Draw three 2-simplexes which intersect nicely and whose union is itself a 2-simplex. (Hint: start with a 2-simplex as the union and subdivide it.)

3. You are asked to draw various 2-dimensional polyhedrons, each as the union of six 2-simplexes. Draw one such that
   (a) no two of the 2-simplexes intersect.
   (b) there is one point common to all the 2-simplexes but no other point is common to any pair.
   (c) the polyhedron is a rectangle together with its interior.

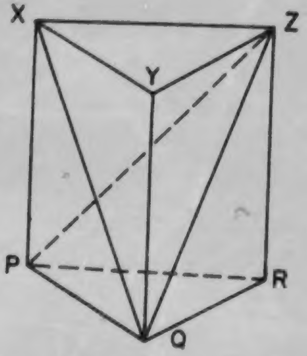
4. The figure on the right represents a polyhedron as the union of 2-simplexes without nice intersections. Draw a similar figure yourself, and then draw in three segments which will make the polyhedron the union of 2-simplexes which intersect nicely.



5. The 2-dimensional figure on the right can be expressed as a union of simplexes with nice intersections in many ways. Draw a similar figure and
   (a) by drawing segments express it as the union of six 2-simplexes without using more vertices.



(b) by adding one vertex near the middle (in another drawing of the figure), express the polyhedron as the union of eight 2-simplexes all, having the point in the middle as one vertex.



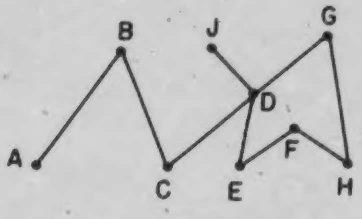
6. (a) List eight 2-simplexes whose union is the surface of the triangular prism on the right. (The figure is like that used earlier.)

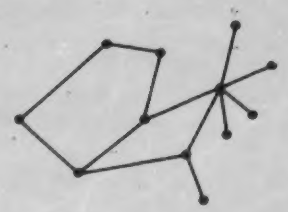
(b) Regarding F as a point in the interior of the prism list eight 3-simplexes (each containing F) whose union is the solid prism.

(c) The figure shows the triangular prism PQRXYZ as the union of three 3-simplexes which intersect nicely. Name them.

10-6. One-Dimensional Polyhedrons

A 1-dimensional polyhedron is the union of a certain number of 1-simplexes (segments). A 1-dimensional polyhedron may be contained in a plane or it may not be. Look at a model of a tetrahedron. The union of the edges is a 1-dimensional polyhedron. It is the union of six 1-simplexes, and does not lie in a plane. We may think of the figures below as representing 1-dimensional polyhedrons that do lie in a plane (the plane of the page).

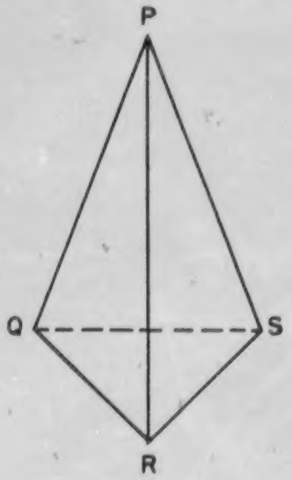




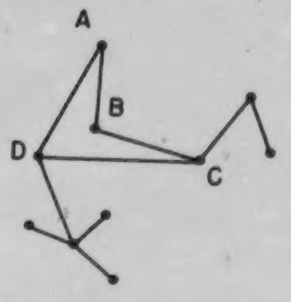
There are two types of 1-dimensional polyhedrons which are of special interest. A polygonal path is a 1-dimensional polyhedron in which the 1-simplexes arranged are in order as follows: There is a first one and there is a last one. Each other 1-simplex of the polygonal path has one vertex in common with the 1-simplex which precedes it, and one vertex in common with the 1-simplex which follows it. There are no extra intersections. The first and last vertices (points) of the polygonal path are called the endpoints.

Neither of the 1-dimensional polyhedrons in the figures above is a polygonal path. But each contains many polygonal paths. The union of (AB), (BC), (CD), (DG) and (GH) is a polygonal path from A to H. The union of (JD) and (DE) is a polygonal path from J to E, and consists of just two 1-simplexes.

In the drawing of a tetrahedron on the right, the union of (PQ), (QR), and (RS) is a polygonal path from P to S (with endpoints P and S). The 1-simplex (PS) is itself a polygonal path from P to S. Consider the 1-dimensional polyhedron which is the union of the edges of the tetrahedron, and find another polygonal path from P to S in it. (Use a model if it helps you see it.) How many such polygonal paths are there from P to S?

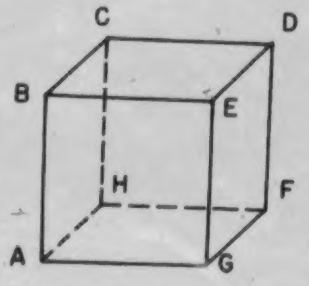


The union of two polygonal paths that have common endpoints is called a simple closed polygon (it is also a simple closed curve). The 1-dimensional polyhedron on the

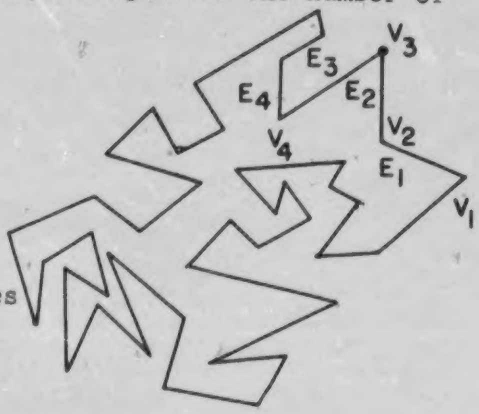


right is not a simple closed polygon, but it contains exactly one simple closed polygon, namely the union of the polygonal paths ABC and ADC which have endpoints A and C in common.

Another way of describing a simple closed polygon is to say that it is a 1-dimensional polyhedron which is in one piece, and has the property that every vertex of it is in exactly two 1-simplexes of it. The simple closed polygon ABCD is then looked on as the union of (AB), (BC), (CD) and (DA).



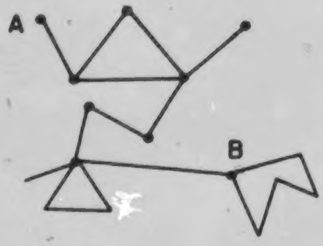
The union of the edges of the cube in the drawing is a 1-dimensional polyhedron. It contains many simple closed polygons. One is the union of (AB), (BE), (EG), and (GA). Another is the union of (AB), (BC), (CD), (DE), (EG), and (GA). **List the vertices, naming at** least two more simple closed polygons containing (BE) and (GA). (Use a model if it helps you see it.)



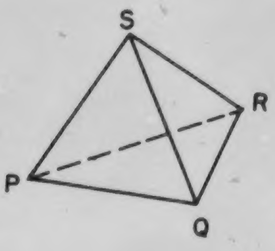
There is one very easy relationship on any simple closed polygon. The number of 1-simplexes (edges) is equal to the number of vertices. Consider the figure on the right. Suppose we start at some vertex. Then we take an edge containing this vertex. Next we take the other vertex contained in this edge and then the other edge containing this second vertex. We may think of numbering the vertices and edges as in the figure. We continue the process. We finish with the other edge which contains our original vertex. We start with a vertex and finish with an edge after having alternated vertices and edges as we go along. Thus the number of vertices is the same as the number of edges.

Exercises 10-6

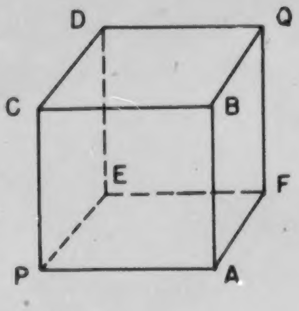
1. The figure on the right represents a 1-dimensional polyhedron. How many polygonal paths does it contain with endpoints A and B? How many simple closed polygons does it contain?



2. (a) The union of the edges of a 3-simplex (solid tetrahedron) contains how many simple closed polygons?
   (b) Name them all.
   (c) Name one that is not contained in a plane. (Use a model if you wish.)



3. Let P and Q be vertices of a cube which are diametrically opposite each other (lower front left and upper back right). Name three polygonal paths from P to Q each of which contains all the vertices of the cube and is in the union of the edges. (Use a model if you wish.)

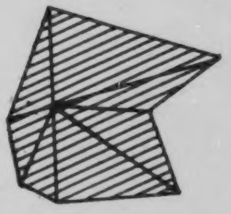


4. Draw a 1-dimensional polyhedron which is the union of seven 1-simplexes and contains no polygonal path consisting of more than two of these simplexes.

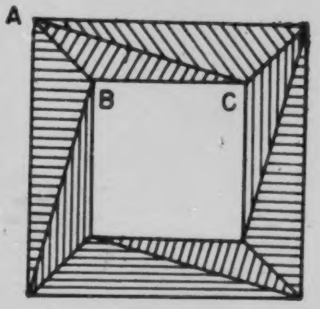
5. Draw a simple closed polygon on the surface of one of your models of a cube which intersects every face and which does not contain any of the vertices of the cube.

10-7. Two-Dimensional Polyhedrons

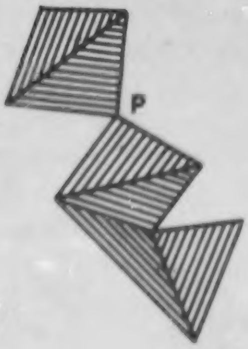
A 2-dimensional polyhedron is a union of 2-simplexes. As stated before, we will agree that the 2-simplexes are to intersect nicely. That is, if two 2-simplexes intersect, then the intersection is either an edge of both, or a vertex of both. There are many 2-dimensional polyhedrons; some are in one plane but many are not in any one plane. The surface of a tetrahedron, for instance, is not in any one plane. Let us first consider a few 2-dimensional polyhedrons in a plane. In drawing 2-simplexes in a plane we shall shade their interiors.



Every 2-dimensional polyhedron in a plane has a boundary in that plane. The boundary is itself a 1-dimensional polyhedron. The boundary may be a simple closed polygon as in the figure on the right. In the figure on the right below, we have indicated a polyhedron as the union of eight 2-simplexes. (ABC) is one of them. The boundary is the union of two simple closed polygons, the inner square and the outer square. These two polygons do not intersect.

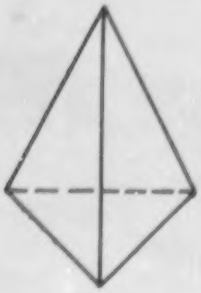


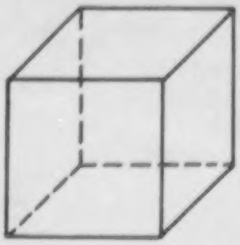
The figure on the right represents a 2-dimensional polyhedron which is the union of six 2-simplexes. The boundary of this polyhedron in the plane is the union of two simple closed polygons which have exactly one vertex of each in common, the point P.

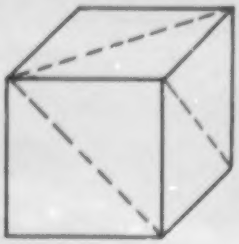


Suppose a 2-dimensional polyhedron in the plane has a boundary which is a simple closed polygon (and nothing else). Then the number of 1-simplexes (edges) of the boundary is equal to the number of 0-simplexes (vertices) of the boundary. You have already seen, in the previous section, why this must be true.

There are many 2-dimensional polyhedrons which are not in any one plane. The surface of a tetrahedron is such a polyhedron. The surface of a cube is another. We have seen that the surface of a cube may be considered to be expressed as the union of 2-simplexes. Here we have some 2-dimensional polyhedrons which are themselves surfaces or boundaries of 3-dimensional polyhedrons. Let us consider these two surfaces, the surface of a tetrahedron and the surface of a cube.







You may look at the drawings above or you may look at some models (or both). Let us count the number of vertices, the number of edges, and the number of faces. The surface of a cube can be considered in at least two different ways. We can think of the faces as being square regions (as in the middle figure), or we may think of each square face as subdivided into two 2-simplexes (as in the figure on the right). We will use F for the number of faces, E for the number of edges and V for the number of vertices. If you are counting from models and do not observe patterns to help you count, it is usually easier to check things off as you go along. That is, mark the objects as you count them.

Let us make up a table of our results.

	F	E	V
Surface of tetrahedron	?	6	?
Surface of cube (square faces)	?	?	8
Surface of cube (two 2-simplexes on each square face)	12	?	?

It is not easy from just these three examples to observe any relationship among these numbers. What we are looking for is a relationship which will be true not only for these 2-dimensional polyhedrons but also for others like these. See if you can discover a relationship which is true in each case.

Exercises 10-7

1. Make up a table as in the text showing F, V, and E for the 2-dimensional polyhedrons mentioned there.

2. (a) Draw a 2-dimensional polyhedron in the plane with the polyhedron the union of ten 2-simplexes such that its boundary is a simple closed polygon.

   (b) Similarly draw another such polyhedron, such that its boundary is the union of three simple closed polygons having exactly one point in common.

   (c) Draw another, such that its boundary is the union of two simple closed polygons which do not intersect.

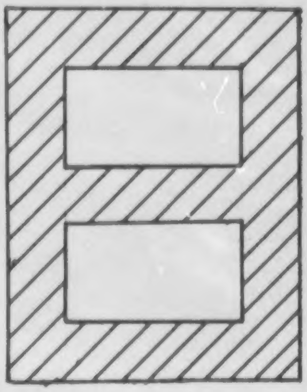
3. Draw a 2-dimensional polyhedron in the plane with the number of edges in the boundary
   (a) equal to the number of vertices.
   (b) one more than the number of vertices.
   (c) two more than the number of vertices.

4. Draw a 2-dimensional polyhedron which is the union of three 2-simplexes, with each pair having one edge in common. Do you think that there exists in the plane a polyhedron which is the union of four 2-simplexes such that each pair has only one edge in common? Does one exist in space?

5. On one of your models of a cube, mark six points, one at the center of each face. Consider each face to be subdivided into four 2-simplexes each having the center point as a vertex. Count F (the number of 2-simplexes), E (the number of 1-simplexes), and V (the number of 0-simplexes) for this subdivision of the whole surface. Keep your answers for later use.

6. Do Problem 5 without using a model and without doing any actual counting. Just figure out how many of each there must be. For instance, there must be 14 vertices, 8 original ones and 6 added ones.

*7. Express the polyhedron on the right as a union of 2-simplexes which intersect nicely (in edges or vertices of each other).



*8. State whether the following statements are true or false. If the statement is not true, correct it so that it is a true statement.

(a) Any 0-simplex (not an endpoint) contained in a given 1-simplex, determines two distinct 1-simplexes whose union is the original 1-simplex, and whose intersection is the given 0-simplex.

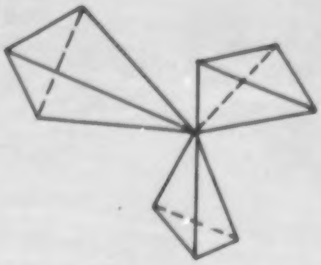
(b) Any 1-simplex (not a boundary) contained in a given 2-simplex, determines two distinct 2-simplexes whose union is the original 2-simplex, and whose intersection is the given 1-simplex.

(c) Any 2-simplex (not a boundary) contained in a given 3-simplex, determines two distinct 3-simplexes whose union is the original 3-simplex, and whose intersection is the given 2-simplex.

10-8. Three-Dimensional Polyhedrons

A 3-simplex is one 3-dimensional polyhedron. A solid cube is another 3-dimensional polyhedron. A 3-dimensional polyhedron is any union of 3-simplexes in which the simplexes of a polyhedron intersect nicely. That is, if two 3-simplexes intersect, the intersection is a 2-dimensional face (2-simplex) of each, or an edge (1-simplex) of each, or a vertex (0-simplex) of each.

Any 3-dimensional polyhedron has a surface (or boundary) in space. This surface is itself a 2-dimensional polyhedron. It is the union of several 2-simplexes (which intersect nicely). The surface of a 3-dimensional polyhedron is represented by the drawing on the right. The surface consists of the surfaces of three tetrahedrons which have exactly one point in common.



The simplest kinds of surfaces of 3-dimensional polyhedrons are called simple surfaces. The surface of a cube and the surface of a 3-simplex are both simple surfaces. There are many others. Any surface of a 3-dimensional polyhedron obtained as follows will be a simple surface. Start with a solid tetrahedron. Fasten to it another, so that the two tetrahedrons have an intersection which is a face of the tetrahedron you are adding. You may continue adding solid tetrahedrons, in any combination or of any size, provided that each tetrahedron added has, with the polyhedron you have already formed, an intersection which is a union of one, two, or three faces of the 3-simplex you are adding. The surface of any polyhedron formed in this way will be a simple surface.

The figure above does not represent a simple surface. Why?

Class activity. Take five models of regular tetrahedrons of edges 3". Put a mark on each of the four faces of one of these. Now fasten each of the others in turn to one of the marked faces. The marked one should be in the middle and you won't see it any more. The surface of the object you have represents a simple surface. You can see how to fasten a few more tetrahedrons to get more and more peculiar looking objects. Suppose it is true that whenever you add a solid tetrahedron the intersection of what you add with what you already have is one face, two faces, or three faces of the one you add. The surface of what you get will be a simple surface.

One can also fasten solid cubes together to get various

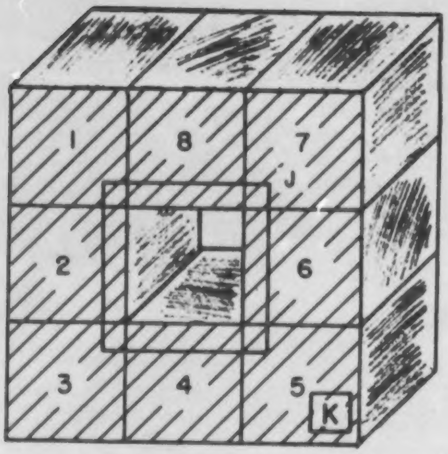
3-dimensional polyhedrons. If you wish them to have simple surfaces, you must follow a rule like the one given before. The cubes must be fastened together in such a way that the intersection of the polyhedron you already have with the cube you are adding is a set which is bounded by a simple closed polygon. For example, the intersection might be a face, or the union of two or more adjacent faces, of the cube you are adding, or the intersection might consist of parts of one or more faces. The important thing is that the intersection is bounded by a simple closed polygon.

Finally we mention an interesting property of simple surfaces. Draw any simple closed polygon on a simple surface. Then this polygon separates the simple surface into exactly two sets, each of which is connected, i.e., is in one piece.

Class activity. On the surface of one of the peculiar 3-dimensional polyhedrons (with simple surface) that you have constructed above, let one student draw any simple closed polygon (the wider the better). It need not be on just one face. Then let another student start coloring somewhere on the surface but away from the polygon. Let him color as much as he can without crossing the polygon. Then let another student start coloring with another color at any previously uncolored place. Color as much as possible but do not cross the polygon. When the second student has colored as much as possible, the whole surface should be colored.

If you do not follow completely the instructions for constructing a polyhedron with a simple surface, you may get a polyhedron whose surface is not simple. Suppose, for instance, you fasten eight cubes together as in the drawing on the next page. The polyhedron looks something like a square doughnut. Note that in fitting the eighth one, the intersection of the one you are adding with what you already have is the union of two faces which are not adjacent. The boundary of the intersection is two simple closed polygons, not just one as it should be.

There are many simple closed polygons on this surface which do not separate it at all. The polygon J does not separate the



surface. The polygon K does. Illustrate this for J and for K by coloring as much as you can of the surface without crossing the polygon.

Exercises 10-8

1. Using a block of wood (with corners sawed off if possible), draw a simple closed polygon on the surface making it intersect most of all of the faces of the solid. Start coloring at some point. Do not cross the polygon. Color as much as you can without crossing the polygon. When you have colored as much as you can, start coloring with a different color on some uncolored portion. Again color as much as you can without crossing the polygon. You should have the whole surface colored when you finish.

2. Go through the same procedure as in Problem 1 but with another 3-dimensional solid. Use one of your models or another block of wood. Make your simple closed polygon as complicated as you wish.

3. How many different kinds of polyhedrons (in terms of their intersections) can you construct from just two cubes of the same size that intersect nicely, i.e., whose intersection is a face, and edge, or a vertex? Illustrate by sketches or models.

4. Take three cubes of the same size. Place them together so that their union has a simple surface. How many different polyhedrons can you construct in this way?

5. Three cubes of the same size are to be placed together so that they intersect nicely.

   (a) Construct at least 5 different polyhedrons formed in this way.

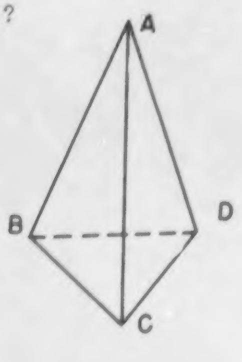
   (b) Do they all have simple surfaces?

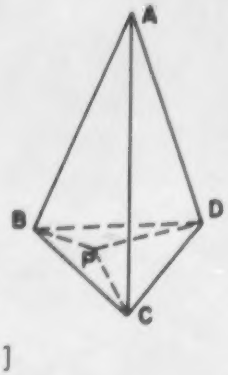
   (c) Could more than 5 such polyhedrons be constructed?

*6. Show that it is possible to fit 7 cubes together so as to form a polyhedron without a simple surface.

10-9. Counting Vertices, Edges, and Faces - Euler's Formula

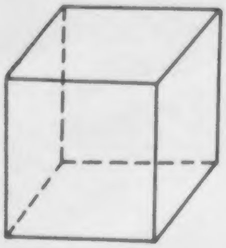
In Section 10-7 you were asked to do some counting. We now look at the problem in another way. A few of you may have discovered a relationship between F, E, and V. Consider the tetrahedron in the figure below. Its surface is a simple surface. What relationship can we find among the number of its vertices, edges, and faces?





There are the same number of edges and faces coming into the point A, three of each. One may see that on the base, there are the same number of vertices and edges. We have two objects left over; the vertex A at the top and the face (BCD) at the bottom. Otherwise we have matched all the edges with vertices and faces. So $V + F - E = 2$. Now let us ask what would be the relationship if one of the faces or the base were broken up into several 2-simplexes. Suppose we had the base broken up into three 2-simplexes by adding one vertex P in the interior of the base. The figure on the right on the previous page illustrates this. Our counting would be the same until we got to the base, and we would be able to match the three new 1-simplexes which contain P with the three new 2-simplexes on the base. We have lost the face which is the base, but we have picked up one new vertex P. Thus, the number of vertices plus the number of 2-simplexes is again two more than the number of 1-simplexes, and $V + F - E = 2$.

Next let us look at a cube. We have a drawing of one on the right. The cube has how many faces? How many edges? How many vertices? Is the sum of the number of vertices and the number of faces two more than the number of edges? Let us see why this must be.



(1) The number of vertices on the top face is the number of edges on the top face.

(2) The number of vertices on the bottom face is the number of edges on the bottom face.

(3) The number of vertical faces is the number of vertical edges.

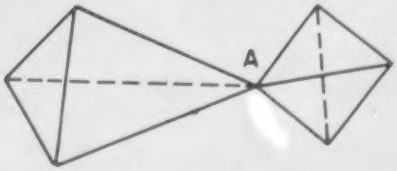
(4) (The number of vertices) + (the number of vertical faces) - (the number of edges) = 0. We have counted all vertices, edges, and faces except the top and bottom faces. Hence,

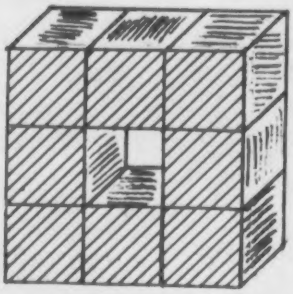
$$V + F - E = 2.$$

What would happen if each face were broken up into two 2-simplexes? For each face of the cube you would now have two 2-simplexes. But for each face you would have one new 1-simplex lying in it. Other things are not changed. Hence V + F - E is again 2.

Suppose we have any simple surface. Then do you suppose that V + F - E = 2? In the exercises you will be asked in several examples to verify this formula, which is known as Euler's formula. Euler (pronounced "Oiler") is the name of a famous mathematician of the early 18th century.

Let us now observe that the formula does not hold in general for surfaces which are not simple. Consider the two examples below.





In the figure above on the left (the union of the two tetrahedrons having exactly the vertex A in common), V + F - E = ? Count and see. Use models of two tetrahedrons if you wish. V + F - E should be 3. On each tetrahedron, separately, the number of faces plus the number of vertices minus the number of edges is 2. But the vertex A would have been counted twice. So V + F is one less than E + 4.

The figure on the right above is supposed to represent the union of eight solid cubes as in the last section. The possible ninth one in the center is missing. Count all the faces (of cubes), edges and vertices which are in the surface. For this figure V + F - E should be 0. (As a starter, V should be 32)

Finally we put the Euler Formula in a more general setting. Suppose we have a simple surface, and it is subdivided into a number (at least three) of non-overlapping pieces. We require that if two of these pieces intersect then the intersection be either one point or a polygonal path. The number E is the number of these intersections of pairs of pieces which are not just points. The number V is the number of points each of which is contained in at least three of these pieces. Then $V + F - E = 2$.

Exercises 10-9

1. Take a cardboard model of a non-regular tetrahedron. In each face add a vertex near the middle. Consider the face as the union of three 2-simplexes so formed. Give the count of the faces, edges, and vertices of the 2-simplexes on the surface. How do the faces, edges, and vertices of this polyhedron compare with those of the polyhedron you get by attaching four regular tetrahedrons to the four faces of a fifth?

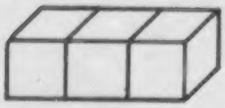
2. Take a model of a cube. Subdivide it as follows. Add one vertex in the middle of each edge. Add one vertex in the middle of each face. Join the new vertex in the middle of each face with the eight other vertices now on that face. You should have eight 2-simplexes on each face. Compute F, V, and E. Do you get $V + F - E = 2$?

3. Make an irregular subdivision of any simple surface into a number of flat pieces. Each piece should have a simple closed polygon as its boundary. Count F, V, and E for this subdivision of the surface.

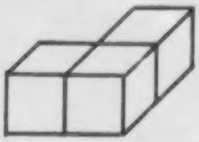
4. Take a cardboard model of a tetrahedron. Mark the midpoint of each edge. In each of the four faces draw the lines joining the midpoints of the edges. In this way each face has been subdivided into four 2-simplexes. Count the number of faces, vertices, and edges of this simple surface and determine the value of $V + F - E$.

5. Three solid cubes can be placed together in two different ways to form the two polyhedrons sketched below. In each case find V, F, and E by counting and then compute V + F - E.

(a)

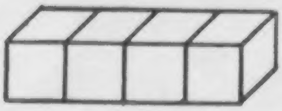


(b)

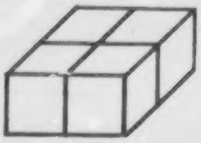


6. Place four solid cubes together as in the sketches below. Find V + F - E in each case.

(a)



(b)



Chapter 11

VOLUMES AND SURFACE AREAS

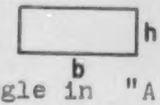
11-1. Areas of Plane Figures

In your earlier work with measurement you studied ways of finding areas of the interiors of various plane figures. This section will largely be a review, but several new geometric figures will be introduced.

You recall that area of a surface is the number of square units contained in it. When we speak of the area of a rectangle, for instance, we will mean the area of the rectangular closed region.

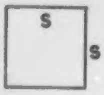
To find the measure of the area of a rectangle you multiplied the measures of the length and width.

Stated as a formula, $A_{\text{▭}} = bh$.



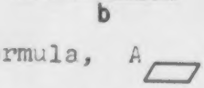
A symbol such as the rectangle in "$A_{\text{▭}}$" will sometimes be used. This identifies further the particular figure which is being discussed.

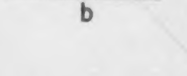
If adjacent sides of a rectangle are congruent, the figure is a square.



Thus, in this case, $A_{\square} = s \cdot s$ or $A_{\square} = s^2$.

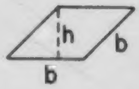
A parallelogram has the same area as a rectangle of the same height and base, as shown in the following figure:





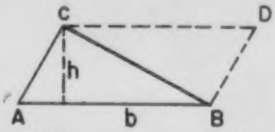
Stated as formula, $A_{\text{▱}} = bh$.

Suppose adjacent sides of the parallelogram are congruent as in the following figure. Such a figure is called a rhombus.



Since it is a parallelogram, its area can be found by the formula $A_{\square} = bh$.

The area of a triangle is found by comparing it with a parallelogram.



In any triangle, ACB, if $\overline{CD}$ is drawn parallel to $\overline{AB}$, and $\overline{BD}$ parallel to $\overline{AC}$, then ABDC is a parallelogram. $\overline{CB}$ separates the interior of the parallelogram into two regions of equal area. The area of the parallelogram is found by multiplying b and h. Thus the area of the triangle can be found by the formula $A_{\Delta} = \frac{1}{2}bh$.

From your study of circles you learned by one or more methods that if r is the measure of the radius of a circle, the measure of the area is found by multiplying the square of r by π. That is,

$$A_{\odot} = \pi r^2$$

Exercises 11-1a

Find the area of each of the following figures after:

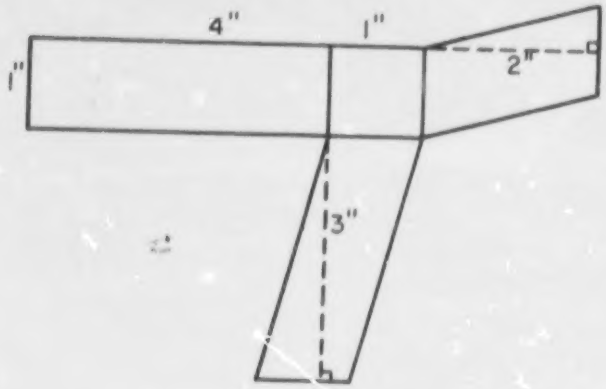
(a) first making a rough drawing of the figure, and

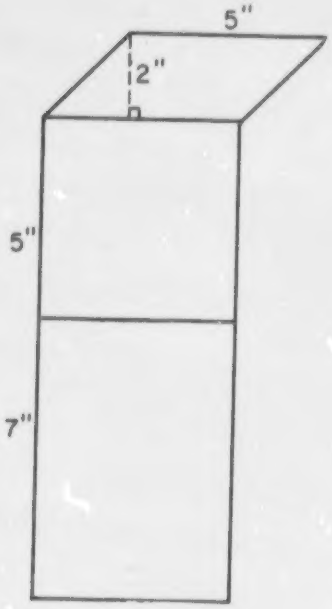
(b) indicating the measurements on the drawing.

	Figure	Measurements
1.	Rectangle ABCD	$\overline{AB}$ is 4" long, $\overline{BC}$ is 12" long.
2.	Rectangle ABCD	$\overline{AB}$ is $\frac{3}{4}$ ft. long, $\overline{AD}$ is 5 ft. long.

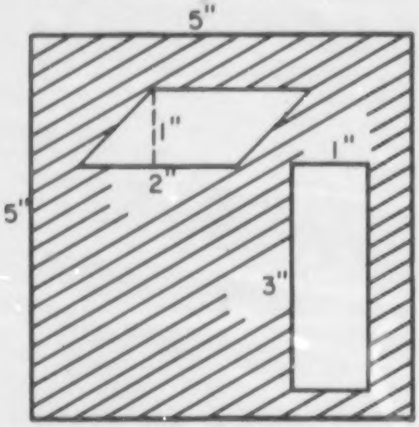
3.	Square ABCD	$\overline{AB}$ is 13 in. long
4.	Square XYZW	$\overline{YZ}$ is $3\frac{1}{2}$ ft. long.
5.	Parallelogram ABCD	$\overline{AB}$ is 16 in. long; the height is 15 in.
6.	Rhombus ABCD	$\overline{CD}$ is 6.5 cm. long; the height is 5 cm.
7.	Rhombus RSTU	$\overline{ST}$ is 5.2 ft. long; height is 4.6 ft.
8.	Right triangle ABC	Angle A is the right angle, and $\overline{AB}$ is 14 cm. long, $\overline{AC}$ is 9.3 cm. long.
9.	Triangle XYZ	The base $\overline{YZ}$ is 38 ft. long, the height $\overline{XW}$ is 37 ft.
10.	A circle	Length of radius is 4.5 in. ($\pi \approx 3.14$)

11. Determine the area of the interior of each of the following plane figures:

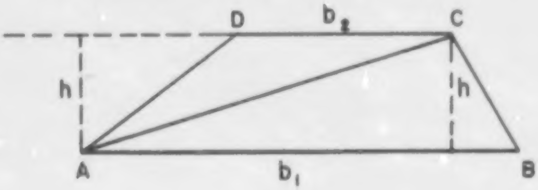




12. Find the area of the shaded portion.



Another geometric figure with which we need to become familiar is a trapezoid, such as ABCD below. A trapezoid is a quadrilateral, only two of whose sides are parallel.



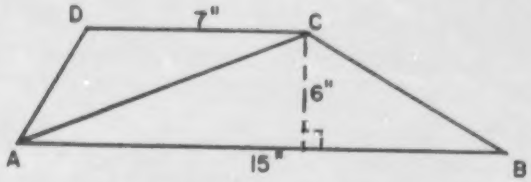
If a diagonal (such as $\overline{AC}$) is drawn, the interior of the trapezoid is separated into two triangular regions. Note that the altitudes shown of both triangles are congruent, but the bases b_1 and b_2, of the two triangles have different measures. The area of the trapezoid is the sum of the areas of the two triangles:

Area of ABCD = Area of ABC + Area of ADC

$$A_{ABCD} = \frac{1}{2} hb_1 + \frac{1}{2} hb_2$$

Notice that the lengths of $\overline{AD}$ and $\overline{BC}$ are not involved in the computation for obtaining the area of the trapezoid.

Example: In the trapezoid ABCD find the area by finding the sum of the areas of the two triangles into which the diagonal $\overline{AC}$ separates it.



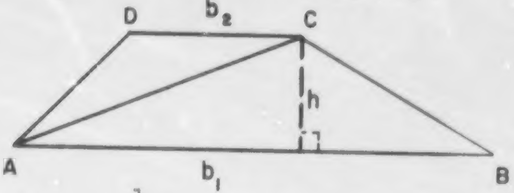
$$A_{\Delta ABC} = \frac{1}{2} \cdot 6 \cdot 15 = 45$$

$$A_{\Delta ADC} = \frac{1}{2} \cdot 6 \cdot 7 = 21$$

$$A_{⏢} = 45 + 21 = 66$$

The area of the trapezoid is 66 square inches.

The above method of finding the area of a trapezoid can be simplified by using the distributive property.



$$A_{ABCD} = \frac{1}{2} h \cdot b_1 + \frac{1}{2} h \cdot b_2$$

Then by the distributive property,

$$A_{ABCD} = \frac{1}{2} h (b_1 + b_2)$$

The formula may also be written:

$$A_{ABCD} = \frac{h}{2}(b_1 + b_2) \quad \text{or} \quad \frac{h(b_1 + b_2)}{2} \quad \text{or} \quad h(\frac{b_1 + b_2}{2})$$

This shows that we can also think of the area of a trapezoid as being obtained by multiplying the measure of the height by the average of the measures of the bases.

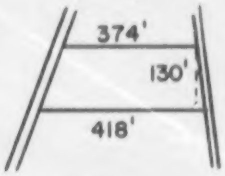
Exercises 11-1b

In Problems 1 to 5, find the area of trapezoids having the given measurements:

	Height	Upper Base	Lower Base
1.	8 in.	6 in.	13 in.
2.	16 in.	35 in.	37 in.
3.	13 cm.	11 cm.	27 cm.
4.	5.4 ft.	9.8 ft.	12.7 ft.
5.	$2\frac{1}{2}$ ft.	$3\frac{1}{4}$ ft.	$6\frac{1}{4}$ ft.

*6. The area of a trapezoid is 696 sq. in. The lengths of the bases are 23 in. and 35 in. Find the height of the trapezoid.

*7. A piece of land between two streets is the shape of a trapezoid.



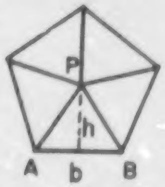
It is to be sold at 30¢ a square foot. Using the measurements given in the figure,

(a) Find the area in square feet.

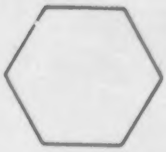
(b) Find the selling price of the land.

Area of a Regular Polygon

(a)



(b)



Recall that a regular polygon is defined to be a polygon whose sides have equal measures and whose angles have equal measures.

Join the center of the regular polygon to each vertex of the polygon. (The center is the point in the interior which is equally distant from the vertices of the polygon and also equally distant from the sides.) If there are n vertices there will be n congruent triangles.

The area of any such regular polygon will be the sum of the areas of the triangles. In Figure (a), for instance, we will first find the area of triangle APB:

$$A_{\Delta} = \frac{1}{2}hb$$

There are five such triangles, so:

$$\begin{aligned} A_{\pentagon} &= \frac{1}{2}hb + \frac{1}{2}hb + \frac{1}{2}hb + \frac{1}{2}hb + \frac{1}{2}hb \\ &= \frac{1}{2}h(b + b + b + b + b) \end{aligned}$$

But $(b + b + b + b + b)$ is the measure of the perimeter of the polygon. Thus $A_{\pentagon} = \frac{1}{2}hp$.

Suppose the regular polygon had ten sides instead of five. Then the lines from the center to the ten vertices would divide the polygon into ten congruent triangles and the area of the polygon would be equal to

$$\frac{1}{2}h(10b) = \frac{1}{2}hp,$$

where p is the perimeter of the polygon.

From this we can see that the area of any regular polygon is equal to

$$\frac{1}{2}hp,$$

where p is its perimeter and h is the altitude of one of the congruent triangles into which the polygon is divided by lines from its center to its vertices.

Exercises 11-1c

Find the areas of the following regular polygons:

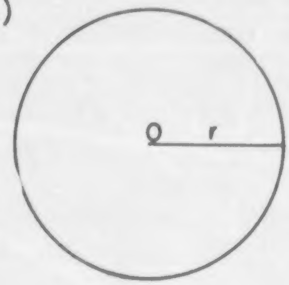
	Kind of Polygon	Length of Perpendicular from Center to Side.	Length of a Side
1.	Hexagon	17.3 inches	20 inches
2.	Pentagon	27.5 inches	40 inches
3.	Octagon	72.5 feet	60 feet
4.	Decagon	30.8 inches	20 inches

Area of a Circle

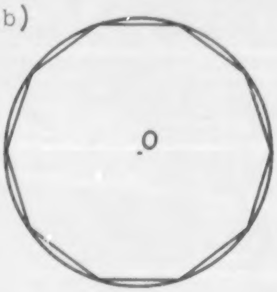
Now we show how the formula we have just found can be used to derive the formula for the area of a circle from the formula for the circumference of a circle.

Consider circle O in the figures below.

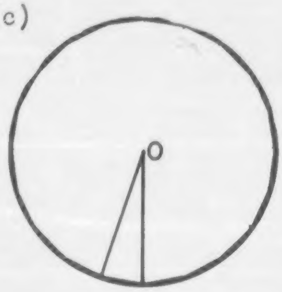
(a)



(b)



(c)



We say that a regular polygon of n sides is inscribed in a circular region if the vertices of the regular polygon are points on the circle. It is clear from the figures above that the more sides the inscribed polygon has the shorter will be the length of each side. Also you will notice that as n gets larger and larger it will be more and more difficult to distinguish between the regular polygon and the circle.

We could say that the area of the interior of the inscribed polygon is approximately equal to the area of the interior of the circle. It will always be less than the area of the circle since there will always be points on the circle that are not vertices of the inscribed regular polygon. Therefore there is always some portion of the area of the circle which is not contained in the interior of an inscribed regular polygon. However, for large values of n, the areas are almost equal. We can think of the area of the interior of the circle as the "upper limit" of the area of the inscribed polygons, that is

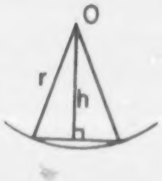
(Area of circle) - (Area of polygon)

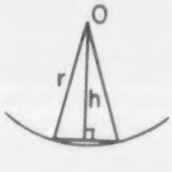
is greater than zero but is very small if the number of sides of the polygon is very large. In fact, by taking the number of sides large enough, the difference may be made as small as you please.

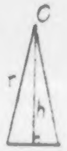
Also, as n becomes larger and larger, the distance from the center of the polygon to a side will become closer and closer to the radius of the circle; likewise the perimeter of the polygon will become closer and closer to the circumference of the circle. We have seen that the number of square units of area in the polygon is $\frac{1}{2}hp$. But we have just observed that when n gets very large h gets close to r and p gets close to $2\pi r$ so we are led to conclude:

If r is the number of linear units in the radius of a circle, and A the number of square units of area in its interior, then

$$A = \frac{1}{2}r(2\pi r)$$
$$A = \pi r^2.$$







Exercises 11-1d

1. Compute the area of the interior of each of the following circles. The measurements in each case are in inches. Express the answer in terms of π.

   (a) $r = 5$
   (b) $r = 10$
   (c) $r = 20$
   (d) $r = 4\frac{1}{2}$
   (e) $d = 30$
   (f) $d = 28$

2. By examining the results of (a), (b), (c) in Problem 1 above tell the effect on the area of a circle if its radius is doubled.

3. (a) BRAINBUSTER. Imagine that you have inscribed a regular polygon of 20 sides in a circle and that you have divided this polygon into 20 congruent triangles by joining its center to each vertex. Show that these triangles can be rearranged into a parallelogram whose height is almost the radius of the circle and the length of whose base is almost one half the circumference of the circle.

   (b) BRAINBUSTER. Imagine that you have circumscribed a regular polygon of n sides (n very large) about a circle O. (This means that each side of the regular polygon contains exactly one point of the circle.) Develop a plausible argument to support the following statement.

   (Area of circle) < (Area of circumscribed polygon)

   Together with our discussion above this would show that

   (Area of inscribed polygon) < (Area of circle) <
   (Area of circumscribed polygon)

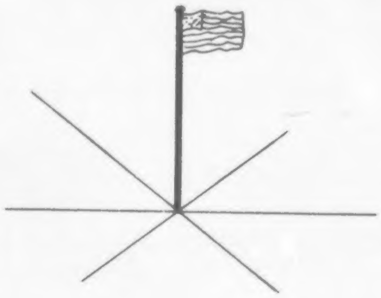
11-2. Planes and Lines

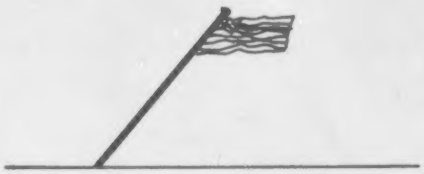
In the following sections of this chapter you will be concerned with surface area and volume of solids. As an aid in this study you will find patterns for models at the end of the chapter. Your teacher will explain to you how these models are to be made. Reference will be made to these models in this section and also throughout the remainder of the chapter.

Before studying this section, make Models 4, 5 and 7; directions found at the end of the chapter. Notice that if you actually measure the indicated segments in the drawings, you will find that what is marked 4", for instance, is not in fact four inches long. But the drawings are to scale, that is, since $1\frac{1}{2}$ is three-eighths of 4, the length marked $1\frac{1}{2}$" is three-eighths of the length marked 4".

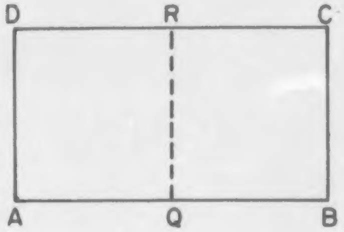
Before going on, let us review briefly some of the simple ideas about planes and lines. You are already somewhat familiar with parallel planes. These are planes which do not have any points in common, that is, whose intersection is the empty set. Such a pair of planes is suggested by the floor and ceiling of some classrooms, or by different floors of an apartment house, or by the covers on a book when the book is closed. Find at least five examples of pairs of parallel planes suggested by things in your classroom.

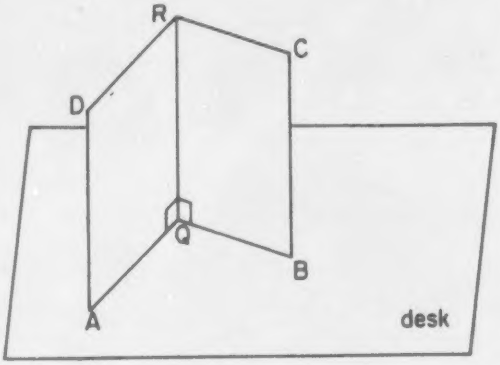
Imagine a flagpole standing in the middle of a level playground, and think of the lines on the playground which run through the base of the pole as shown. What relation does there appear to be between the line represented by the flagpole and these lines drawn on the playground? Our experience certainly suggests that the pole is perpendicular to each of these lines. In fact, if it





were not, then from certain positions the pole would appear like this, which is not at all in accord with our observation. We describe this relationship by saying that the pole is perpendicular to the playground. In general a line which meets a plane in a point A is said to be *perpendicular to the plane* if the line is perpendicular to every line in the plane through A. If a segment lies on a line perpendicular to a plane, we will say that the segment is perpendicular to the plane.





Now try the following simple experiment. Take a piece of notebook paper as shown and fold it over so $\overline{AD}$ falls on $\overline{BC}$. The crease you have made is represented by the dotted segment $\overline{QR}$. Then $\angle AQR$ and $\angle BQR$ are both right angles. How do you know? Now take the paper and set it on your desk as shown, in the position of a partly opened book, so that segments $\overline{AQ}$ and $\overline{BQ}$ lie on the plane of the desk top. Would you agree that $\overline{QR}$ is now perpendicular to the desk top? If so, notice that you have found a line perpendicular to a plane by making it perpendicular to just two different lines in the plane. This illustrates the following property of perpendiculars.

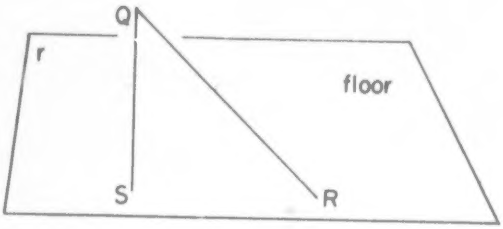
Property 1. *If a line is perpendicular to two distinct intersecting lines in a plane, it is perpendicular to the plane*.

If you help to put up a Christmas tree, check to see whether or not it is perpendicular to the floor by seeing if it is perpendicular as viewed from two different points. If these two points and the tree are not in the same line, the tree is perpendicular from all points of view. This is an application of Property 1.

As another example, examine Model 5 and look at one of the segments which connects a vertex of one hexagonal end with a vertex of the other. As you see, this segment is a part of two rectangles. It is therefore perpendicular to two segments in each hexagon. By Property 1 the segment is therefore perpendicular to the planes of both hexagons. Examine Model 4 similarly and satisfy yourself in the same way that every edge of the solid is perpendicular to the planes of two of its rectangular faces. Notice the line where two walls of your classroom meet. What relation does it have to the planes of the ceiling and floor?

Examine Model 7 to satisfy yourself that the result actually applied to this also.

Now try another experiment. Tie one end of a string to some convenient point Q in your classroom which has a clear space below it. If nothing else is available tie it to a yardstick placed over the back of a couple of chairs, and have someome hold the ends so they won't move. Now select a point R on the floor and notice how much string it takes to join Q to R. By varying R try to find the point S on the floor which requires the least amount of string. When you have located the point S, notice the position $\overline{QS}$ of the string. What relation does it seem to have to the floor? Would you agree with the following statement?



Property 2. The shortest segment from a point Q outside a plane r to the plane r is the segment perpendicular to that plane.

This shortest distance is called the distance from Q to r.

Imagine now several nails in the ceiling of your room, to each of which is attached a string. In each case the string is then attached to the nearest point of the floor as in our experiment above. What do you know about the lengths of the different strings? Will they be all the same? This illustrates the following fact:

Property 3. If two planes are parallel, the (perpendicular) distances from different points of one plane to the other plane are all the same.

The constant distance in Property 3 is called the distance between the parallel planes. Actually the segments involved in Property 3 are perpendicular to both planes. We have already noticed this for the lateral edges of a right prism.

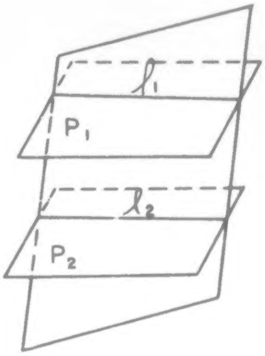
Exercises 11-2

1. Give five examples of pairs of parallel planes with lines perpendicular to both planes in each example.

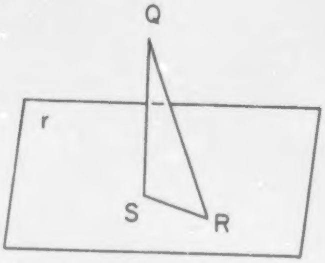
2. Examine Models 4 and 7. Note the sets of parallel planes and the distance between them.

3. Make Models 9 and 10. Note the sets of parallel planes and the distances between them.

4. If two parallel planes, P_1 and P_2, are intersected by a plane r in lines ℓ_1 and ℓ_2, then ℓ_1 must be parallel to ℓ_2. Explain why this is true.



5. We actually could have proved Property 2 instead of observing it by experiment. Give the reasons in the following proof.



Let S be the point r so that $\overline{QS}$ is perpendicular to r. Draw segment $\overline{SR}$.

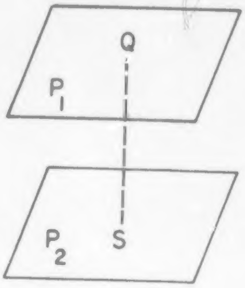
(a) $\angle QSR$ is a right angle. Why?

(b) $\overline{QR}$ is the hypotenuse of a right triangle. Why?

(c) $\overline{QR}$ is longer than $\overline{QS}$. Why?

But since R was any point of r except S, this shows that $\overline{QS}$ is the shortest segment.

6. BRAINBUSTER. A segment $\overline{QS}$ has its ends on the parallel planes P_1 and P_2. If $\overline{QS}$ is perpendicular to P_2, prove it must also be perpendicular to P_1. Hint: Draw two planes through $\overline{QS}$.



11-3. Right Prisms

Since you have already studied some examples of right prisms, this section will be in the nature of a review. Do you remember what kind of a figure is a right prism? Let us review its description.

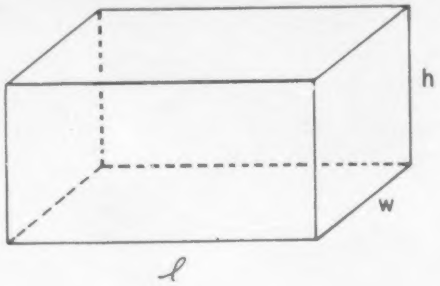
Imagine two congruent polygons so placed in parallel planes that when the segments are drawn joining corresponding vertices of the polygons, the quadrilaterals formed are all rectangles. These rectangles and the original polygons determine closed regions. The union of these closed regions is called a right prism. The segments are its edges, and the points where two or more edges meet are vertices. The rectangular closed regions are called lateral faces (or faces). The original polygonal closed regions are called the bases. The segments joining corresponding vertices of the two bases are called lateral edges. It is well to point out that the edges are perpendicular to the bases, and that, in fact, if the edges are perpendicular to the bases, the faces are automatically rectangles.

A right prism is triangular, rectangular, hexagonal, and so on, according to the shape of its bases. Consider Models 1, 2, 3, 4, 5, 6, and 7 (making those of this set which you have not already made). These are examples of right prisms.

You remember how to find the surface area and volume of right prisms. The surface area is the sum of the areas of the bases and faces. The volume is the product of the measure of the surface area of one base and the measure of the altitude.

Rectangular Right Prisms

One right prism with which you are rather familiar is the rectangular right prism. A good example is a cereal box. The figure below represents such a prism.



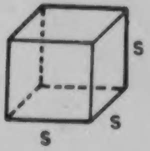
We shall let the measure of its length, width, and height be represented by ℓ, w and h, respectively. Furthermore, we shall let S represent the measure of its surface area and V represent the measure of its volume. Recall the following formulas:

$S = 2(\ell w + \ell h + wh)$

$V = \ell wh$ or $V = Bh$ where B is the measure of the area of the base $(B = \ell w)$. You will notice that in a prism of this type all faces are rectangles, so any pair of parallel faces can be considered as the bases. Can you state these formulas in words? Try it.

Cubes

A cube is a special case of the rectangular right prism in that all of its edges are congruent. Let us designate the measure of its edges by s as in the figure below. Study again Model 1.



Since a cube is a right prism its surface area and volume are obtained in the same manner as you used for the rectangular right prism. The formulas, however, can be shortened since, for a cube,

$$\ell = w = h = s.$$

The formula for surface area may be developed as follows:

$$S = 2\ (\ell w + \ell h + wh)$$
$$S = 2\ (s \cdot s + s \cdot s + s \cdot s)$$
$$S = 2\ (s^2 + s^2 + s^2)$$
$$S = 2\ (3s^2)$$
$$S = 6s^2$$

The formula for the volume of a cube may be developed in a similar manner as follows:

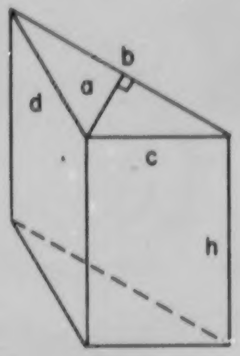
$$V = \ell wh$$
$$V = s \cdot s \cdot s$$
$$V = s^3$$

As an example consider a cube having the measure of its edges 2. Since $s = 2$, then

$$S = 6 \cdot 2^2 = 6 \cdot 4 = 24,$$
$$V = 2^3 = 8.$$

Triangular Right Prisms

This is another prism which you have studied. Again study Model 6. The bases of Model 6 are right triangular regions; however, the bases of a triangular right prism may be any type of triangular region. Consider the figure below. Let the measures of the edges and one altitude of the triangular bases be b, c, d, and a respectively. Also, let the measure of the lateral edges of the prism be h. Now let us develop the formulas for surface area and volume of this type of prism.



Since the area of the surface is the sum of the areas of the bases and the faces, we have:

$$S = 2(\tfrac{1}{2}ab) + (hb + hc + hd)$$
$$S = ab + h(b + c + d)$$
$$S = ab + hp,$$

where p is the measure of the perimeter of the triangular base.

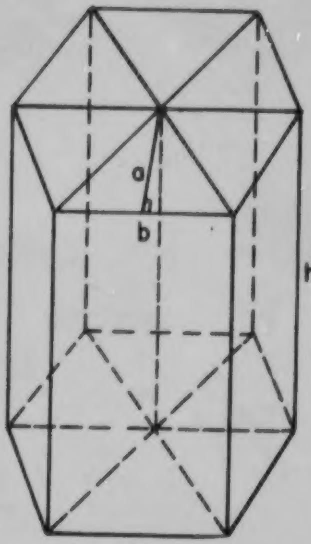
You have used the volume formula before, but it is given below by way of review.

$$V = \frac{1}{2}abh \quad \text{or} \quad V = Bh,$$

where B is the measure of the area of the base.

Hexagonal Right Prisms

You are also familiar with this prism. Study your Model 5. The bases may be any six-sided polygonal regions; however, we will consider only regular hexagons. Consider the following figure.



Let the measures of the lateral edges be h, the edges of the bases be b and the altitudes of the triangles into which the bases are divided be a. The formulas for S and V are now as follows:

$$S = ap + hp$$

where, as you remember, p is the measure of the perimeter of the hexagon.

$$V = \frac{1}{2} aph \quad \text{or} \quad V = Bh,$$

where B is the measure of the area of the base.

Right Circular Cylinders

A right circular cylinder, which you have studied before, is not a prism. It is being introduced, however, because the formula for finding its volume has the same general form as does the right prism. That is: $V = Bh$, where B is the measure of the base.

Recall that $B = \pi r^2$, where r is the measure of the radius of the circular base. The formula then is:

$$V = \pi r^2 h.$$

The formula for the area of the surface will be remembered as:

$$S = 2\pi rh + 2\pi r^2$$

which may be written as $S = 2\pi r(h + r)$.

Exercises 11-3

1. Compute the volume of each right rectangular prism whose measures are as follows:
   (a) $\ell = 1$, $w = 2$, $h = 2$
   (b) $\ell = 2\frac{1}{2}$, $w = 2$, $h = 2$
   (c) $\ell = 1\frac{1}{2}$, $w = 1\frac{1}{2}$, $h = 2$
2. Calculate the surface area of each right rectangular prism of Problem 1.
3. Find the surface area and volume for Model 6.
4. Suppose the base of a right rectangular prism is left unchanged and the measure of its lateral edge doubled, what is the effect on the volume? What is the effect on the sum of the areas of the lateral faces?
5. Suppose ℓ and w of a right rectangular prism are each doubled and the lateral edge left unchanged, what is the effect on the volume? What is the effect on the sum of the areas of the lateral faces?
6. If each of ℓ, w, h are doubled for a rectangular prism, what is the effect on the volume? What is the effect on the sum of the areas of the lateral faces? On the surface area?
7. Construct Model 8. Find the area of its surface and its volume.

8. (a) Refer to patterns for Models 4, 5, 7, 8 and find the perimeters of the bases.
   (b) Are these perimeters all equal to each other?
   (c) Find the volumes of these four models. Use your ruler to find any additional measurements which you need.
   (d) Are the volumes equal?
   (e) List the models in the order of the measures of their volumes from smallest to largest.
   (f) On the basis of your experience in this problem, what conjecture ("conjecture" is a big word for what we hope is an intelligent guess) would you make about the area of the interior of a circle as compared with those of polygons whose perimeters equal the circumferences of the circle?

9. (a) When you computed the volumes of Models 4 and 6, did you find them equal?
   (b) Check (a) by filling one with salt and pouring it into the other.
   (c) Find the perimeters of the bases of these models. Are the perimeters equal?

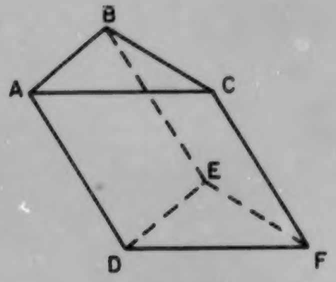
11-4. Oblique Prisms

Now that we have reviewed right prisms, we will study general prisms of which right prisms and oblique prisms are special cases. Models 9, 10, 11, and 12 are examples of oblique prisms. The description of oblique prisms is quite similar to the one you studied in Section 11-3 of right prisms. An oblique prism may be described as follows.

Again, consider two congruent polygons. Imagine them so placed in parallel planes that when the segments are drawn joining corresponding vertices of the polygons the quadrilaterals formed are all parallelograms, of which, at least two must be non-rectangular. This means that the lateral edges are not perpendicular to the bases. These parallelograms and original polygons determine closed regions. The union of these closed regions is called an *oblique prism*. The segments are its edges, and the points where two or more edges meet are vertices. The closed regions formed by the parallelograms are called *lateral faces* (or faces). The original polygonal closed regions are called *bases*. The segments joining corresponding vertices of the two bases are called *lateral edges*.

Wherein does the above description differ from the one for the right prism?

An oblique prism may be illustrated by the following figure. The triangles ABC and DEF, in parallel planes, are the bases.



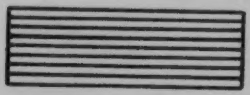
The parallelograms ABED, ACFD, and CBEF are the lateral faces. The lateral edges are $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. Models 9, 10, 11, and 12 will help you to understand this better. In particular, compare Models 6 and 11 by pointing out the bases, lateral faces and lateral edges.

Now do the same for Models 7 and 9. In these last cases did you have any difficulty identifying the bases? How did you decide? The difficulty here illustrates an interesting property of Models

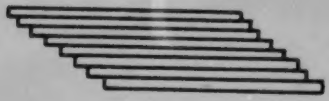
4, 7, 9, 10. In these models all faces are parallelograms. (Recall that a rectangle is a special case of a parallelogram.) In these figures any pair of opposite faces may be considered the bases and the other faces are then the lateral faces. Such figures can really be thought of as prisms in three ways. Because their faces are all parallelograms, such prisms are given the mouth-filling name parallelepipeds. The rectangular prisms which you studied earlier are the special parallelepipeds where all the faces are rectangles.

As in the study of the right prisms we are interested in finding the surface area and volume of oblique prisms. There is no problem in finding the surface area, since this is obtained by finding the sum of the areas of the bases and lateral faces. Of course, in finding the areas of the lateral faces, you will be finding the areas of the parallelograms rather than rectangles.

Now let us consider the volume of an oblique prism. This will require a bit more study. As a beginning, let us consider a stack of rectangular cards which are congruent. You may make such a stack or use a deck of playing cards. When you have the cards stacked so that all adjacent cards fit exactly you have an illustration of a right prism similar to the following illustration in cross section.



Now push the cards a bit so that the deck will have the following appearance in cross section.



You now have an illustration of an oblique prism. Of course it is not a perfect prism due to the thickness of the cards. The cards no longer fit smoothly. You can easily feel the effect by running your fingernail over the edges, and it can be apparent to the eye also if the stack of cards gets far out of the vertical. Still the irregularities seem to be rather small, especially if we imagine we have very thin cards, perhaps made of tissue paper.

Next let us consider a similarity between the two stacks illustrated above. You note that the bases are congruent and the distances between the bases are equal. In view of this discussion it would seem that we have the basis for making the following conjecture.

Conjecture: If two prisms have congruent bases and equal heights, they have equal volumes.

To test this conjecture look at Models 6, 11, and 12. Do they appear to have congruent bases? Do they have equal heights? For this it may help to stand them on their bases and lay a ruler across their upper bases to see if it seems level. Do you agree that these models have congruent bases and equal heights? Now fill Model 6 with salt and pour into Model 11. Did you have too much salt or not enough, or did it seem to be just right? (This sounds like the three bears!) Do your results on this experiment confirm the conjecture above?

Carry out the same experiment with Models 7, 9, and 10. (For this experiment treat the small parallelograms as the bases since otherwise you do not get congruent bases.) Does the result confirm the conjecture?

Since the conjecture seems to be borne out in practice, we will list it now as a property.

Property 4. If two prisms have congruent bases and equal heights, they have equal volumes.

From Property 4 the volume of any prism is the same as that of a right prism whose base is congruent to the given one and having the same height. But since we know how to find the volume of the right prism, we obtain at once the following formula:

> The number of cubic units of volume in any prism is obtained from the formula
>
> $$V = Bh$$

where B is the number of square units of area in its base and h the number of linear units in its height.

For example, the base of Model 11 is a right triangle with sides having lengths approximately 2 inches and $2\frac{1}{4}$ inches. Check these measurements on your model. The number B of square inches in the area of the base is therefore

$$B = \frac{1}{2}(2)(2\frac{1}{4}) = \frac{9}{4}$$

Thus the area of the base is $\frac{9}{4}$ sq. in. Why? You should find the height is 4 in. (Note this is not the same as the length of the lateral edge which is about $4\frac{1}{8}$ in.) Thus $h = 4$.

$$V = (\frac{9}{4})4 = 9$$

and the volume is 9 cu. in.

Exercises 11-4

1. Check the accuracy of the last calculation by taking your cubic inch measure, Model 1, and see if 9 fillings of it will just fill Model 11.
2. Is a lateral edge of a right prism an altitude of the prism? Why?
3. Is a lateral edge of an oblique prism an altitude of the prism? Why?
4. In finding the volume of an oblique prism a student accidentally used the length of a lateral edge in place of the height of the prism. If he made no other errors, was his answer too large or too small?

5. Must all the lateral edges of a prism be congruent? Why or why not?

6. Two faces of a prism are called adjacent if they have a lateral edge in common. Show that if two adjacent faces of a prism are rectangles, the prism is a right prism.

7. Show that if a lateral edge of a prism is congruent to its altitude, the prism is a right prism.

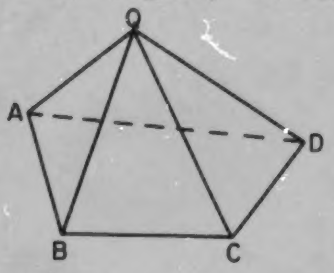
8. If the altitude of a prism is doubled, its base unaltered and all angles unchanged, how does this affect the volume?

9. If all edges of a rectangular prism are doubled and its shape left unchanged, how is the volume affected?

11-5. Pyramids

Make and examine carefully the five Models 13, 14, 15, 16, and 17. These are examples of pyramids. What common property do you observe of these five models?

You should see in each case a figure obtained by joining the vertices of a polygon to a point not in the plane of the polygon, thus forming triangles. The pyramid consists of the closed triangular regions and the closed region of the original polygon. The closed region of the original polygon is called the base of the pyramid and the other faces its lateral faces. The point to which the vertices of the polygon are joined we shall call the apex of the pyramid. (Many books call this the vertex of the pyramid, but we have chosen the term apex since each corner of the polygon is also called a vertex.) The edges meeting at the apex are called lateral edges. For example, in the figure, the base is the interior of quadrilateral ABCD; the lateral faces are the closed regions of the triangles ABQ, BCQ, CDQ, and DAQ; the lateral edges are $\overline{AQ}$, $\overline{BQ}$, $\overline{CQ}$, $\overline{DQ}$; and the apex is Q.



Point out the bases, lateral faces, lateral edges, and apex on each of the Models 14 and 16.

Notice that in Models 13, 15, and 16 the bases are closed square regions. These are called square pyramids. Similarly Model 14 is a hexagonal pyramid. What kind of a pyramid is Model 17? Why?

Although there probably was no argument about the answer to the last question, there might be disagreement over identifying the base. All the faces are triangular closed regions, so how do we distinguish which one is the base? The answer of course is that we can't. Any one of the four faces can be considered as the base, so this figure can be looked at as a triangular pyramid in four different ways. (Compare the case of the parallelpiped which could be considered a prism in three ways.) Because it has just four faces this figure is generally called a tetrahedron. A tetrahedron with its interior is sometimes called a 3-simplex, this was discussed in detail in Chapter 10.

Now look again at the five pyramid models. In each case imagine the segment drawn from the apex perpendicular to the plane of the base. This segment is called the altitude and the length of the altitude is the height of the pyramid. Compare the heights of Models 13, 14, 15, and 16. Laying a ruler across them may help in estimating heights. Do you find the models have equal heights? Model 17 has four heights, depending upon which face is taken as the base. Take the smallest triangular regions as the base and compare the height with that of the other models. Do all five of these models seem to have the same height?

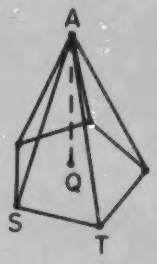
It is not always easy to imagine just where the foot of the altitude will be for a pyramid. In one of the models the altitude coincides with one of the lateral edges, so the foot of the altitude is a vertex of the base. Find the model and the edge. In another model the foot of the altitude is entirely outside the base. Which model? For the other three the foot of the altitude is somewhere in the interior of the base.

The most symmetrical pyramids are called regular pyramids. To be regular, a pyramid must meet two conditions. First, its base must be the closed region of a regular polygon. (A regular polygon is one whose sides are congruent and whose angles are congruent.) Which of the models meet this first condition? Second, the foot of the altitude must be at the center of this regular polygon. Which of the models appear to be regular pyramids?

It is shown in the problems below that the second condition is really the same as saying that the lateral edges all have equal lengths, a fact much easier to recognize by looking at the model.

Exercises 11-5

1. Look at the figure. It is supposed to show a regular pentagonal pyramid with apex A and altitude $\overline{AQ}$. Since Q is the center of the pentagon, it is the same distance from S and from T. Suppose $\overline{AQ}$ is 4 inches long and $\overline{QT}$ and $\overline{QS}$ are each 3 inches long.



   (a) How can you find the lengths of $\overline{AS}$ and $\overline{AT}$?

   (b) What are these lengths?

   (c) Do $\overline{AS}$ and $\overline{AT}$ have equal lengths?

   (d) Is triangle AST isosceles?

   (e) Can the reasoning above be used to show that all five of the lateral edges have the same length?

2. (a) Does the reasoning in the last problem depend on the fact that the base is a pentagonal region or would it work for any regular polygonal region?

   (b) Does the reasoning depend on the particular lengths given, or would it apply to any lengths?

   (c) Complete the following statement:
   If a pyramid is regular then its ________________ are all congruent.

3. Look again at the figure of Problem 1, with the base a regular pentagonal region, but this time suppose we know that the lateral edges all have the same lengths but do not know where the foot Q of the altitude is located. To be definite, suppose the height of the prism (i.e. length of $\overline{AQ}$) is 12 inches, and that each of the lateral edges $\overline{AS}$ and $\overline{AT}$ is 13 inches long.

   (a) How can you find the lengths of $\overline{QS}$ and $\overline{QT}$?

   (b) What are these lengths?

   (c) Are they equal?

   (d) Can this reasoning be used to show that the distances from Q to all five vertices of the polygon are equal?

   (e) Does this show Q is the center of the regular polygon?

   (f) Is the pyramid a regular pyramid?

4. (a) Does the reasoning of Problem 3 depend on the particular measurements and the fact the base is a pentagon?

   (b) If not, complete the following statement:

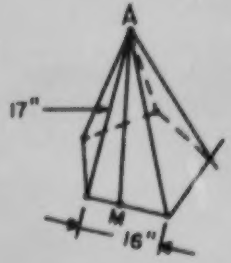
   If, in a pyramid with the closed region of a regular polygon as base, the ________________ are all equal in length, then the pyramid is ________________.

5. Construct a model of a tetrahedron in which all four faces are equilateral triangular regions. Such a figure is called a regular tetrahedron.

6. (a) How many altitudes does a regular tetrahedron have?

   (b) These altitudes of a regular tetrahedron are __________.

7. The base of a regular pentagonal pyramid is 16 in. on a side. If the lateral edges of the pyramid are each 17 in., find the lateral area of the pyramid (the sum of the areas of all five lateral faces). Hint: Draw segment $\overline{AM}$ from apex A to a midpoint of a side of the pentagon. This is the altitude of this triangular face. Its measurement is called the slant height of the regular pyramid. In the figure



$$(AM)^2 + 8^2 = 17^2$$

or,

$$s^2 + 8^2 = 17^2$$

where s designates the slant height.

8. A regular square pyramid has a base which is 10 inches on a side. Its slant height (see problem above) is 12 inches.

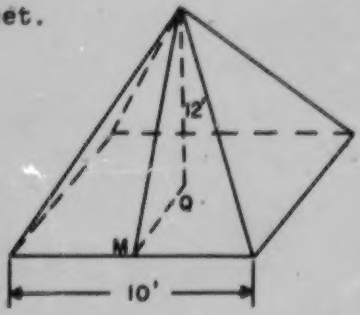
   (a) Find its total area (sum of area of lateral faces and the base).

   (b) Find the lengths of the lateral edges.

9. The base of a regular square pyramid is 10 feet on a side. The altitude of the pyramid is 12 feet.

   (a) Find the total area.

   (b) Find the lengths of the lateral edges. Hint: How far is it from Q to M? Use this to find the slant height.



11-6. Volumes of Pyramids

Can we do anything now about finding volumes of pyramids?

In the section on prisms we found it useful to consider models made up of stacks of cards. Perhaps you and your classmates would like to make a similar model for a pyramid. If so, get some heavy cardboard (such as grocery cartons) and make a series of square pieces to pile on each other. As a suggestion make the bottom one 6 inches on a side, the next one $5\frac{7}{8}$ inches on a side, etc., going down by $\frac{1}{8}$ inch each time. Theoretically you will have 48 square pieces, but actually you will have to omit the very top ones as they get too small to work with. However, you should be able to go up at least to the 1 inch by 1 inch square. To avoid having the square pieces fall apart when you move them, make a hole in the center of each one and run a cord through them, preferably an elastic cord to hold them firmly together.

If you want a larger model, start with a square one foot on a side. This will take twice as many layers and eight times as much cardboard. A deluxe model might be made by cutting the square regions out of $\frac{1}{8}$ inch masonite or something similar in your wood shop. The larger model would take a little over 32 square feet of material, the smaller a little more than 4 square feet.

Such a model should convincingly remind you of a square pyramid, though of course there are irregularities at the edges as in the case of the prism. By shoving the square pieces around you can make this model assume approximate shapes of all kinds of square pyramids. When the square pieces are piled up with the center holes directly above each other it appears as a regular pyramid like our Model 13. By pushing it to one side the apex is no longer above the center of the base. You can very probably push it far enough so that the apex is over a corner of the base as in Model 15, and possibly even into the position of Model 16 where the perpendicular from the apex is outside the base.

In all this moving around we clearly have not changed the base of the pyramid or its height, which is after all just the thickness of our stack of square pieces. Moreover, we have not changed the amount of cardboard in the pile. It looks like a good guess then, that any two pyramids with congruent bases and congruent altitudes have equal volumes.

Let us try this out on Models 13, 15, and 16 which clearly have congruent square bases and whose heights are the same, as we saw earlier. Fill Model 13 with salt and try emptying it into Model 15,and then into 16. Do your results confirm the guess above?

On the basis of this experiment and the evidence of our cardboard model we write the following property:

Property 5. If two pyramids have congruent bases and congruent altitudes they have equal volumes.

To find what the actual volume of a pyramid is however, we must eventually compare it with some figure whose volume we know. As an experiment take Model 13, the regular square pyramid and Model 4, the rectangular right prism. How do the bases of these two models compare (if we take the small square as the base for Model 4)? How do their heights compare? Do you agree they have congruent bases and equal heights? The interior of Model 4 is clearly larger than the interior of Model 13, but how much larger? Fill Model 13 with salt and pour it into Model 4. Keep on doing this until Model 4 is full. According to your results the interior of Model 4 is how many times that of Model 13?

Repeat the experiment with Model 14 and Model 5. Did you get the same multiple in this case? Make a third trial with Model 17 and Model 6. On the basis of these experiments do you agree with the following property?

Property 6. The volume of a pyramid is one third that of a prism whose base is congruent to the base of the prism and whose height is the same as that of the prism.

Since we know how to find the volume of a prism, this leads at once to a formula for finding the volume of any pyramid:

$$V = \frac{1}{3} Bh$$

where B stands for the number of square units of area in the base and h the number of linear units in the height.

Exercises 11-6

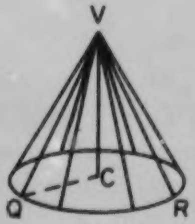
1. Find the volume of the pyramids, the measurements of whose bases and heights are as follows:
   (a) area of base = 12 square inches, height = 7 inches.
   (b) area of base = 100 sq. cm., height = 24 cm.
   (c) area of base = 14,400 sq. ft., height = 60 ft.
2. Model 13 has a square base of $1\frac{1}{2}$ inches on a side and a height of 4 inches. Check these measurements with your model. Then find the volume of Model 13.
3. What is the height of a pyramid whose volume is 324 cu. m. and whose base is a square, 9 m. on a side?
4. The Pyramid of Cheops in Egypt is 480 ft. high, and its square base is 720 ft. on a side. How many cu. ft. of stone were used to build it? (Assume that the pyramid was solid.) How many cu. yards?

*5. Find the total surface area of the regular triangular pyramid whose lateral edge is 12 inches.

*6. The side of the square base of a pyramid is doubled. The height of the pyramid is halved. How is the volume affected?

11-7. *Cones*

Anyone who has eaten an ice cream cone has at least a rough idea of the figure called a cone, or more strictly a right circular cone. Let a circle be drawn as shown below, with center C, and let V be a point not in the plane of the circle so that segment $\overline{VC}$ is perpendicular to this plane.

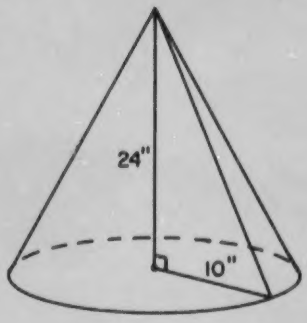


If all the segments from V were drawn to the points of the circle, the union of all these segments, together with the closed circular region, forms a *right circular cone*. The closed circular region is called the *base* of the cone, and the union of the segments is its *lateral surface*. The point V is called the *vertex* of the cone. In the description *right circular cone*, the word circular indicates that the base is the closed circular region, and the word *right* means $\overline{VC}$ is perpendicular to the plane of the circle. Here we consider only right circular cones, and when the word "cone" is used it will mean this type.

Segment $\overline{VC}$ is called the altitude of the cone, and the length of this segment is the height of the cone. If Q is a point of the circle, what kind of triangle is VCQ? Why? If you know the height of the cone and the radius of its base can you find the length of $\overline{VQ}$? How? If R is another point of the circle, do $\overline{VQ}$ and $\overline{VR}$ have the same length? This constant distance from vertex V to the different points of the circle is called the *slant height* of the cone.

If h is the number of linear units in the height of the cone, r the number of linear units in the radius, and s the number of linear units in the slant height, write an equation relating h, r, and s. If you know any two of these numbers can you find the third one from this equation?

As an example, suppose the radius of the base of a cone is 10 in. and the height is 24 in. What is the slant height of the cone? Did you find the slant height to be 26 in.?



Make and examine Model 18. Point out the base, the vertex, and the lateral surface. Approximately what is the slant height? Do you find it is about $4\frac{1}{8}$ inches? Do you find the radius a little less than an inch? Writing these as decimals and rounding to one decimal place, we may take the slant height as 4.1 inches and the radius as 0.9 inches. What is the height of the model? It should be a nice counting number.

How can we find the volume of a cone? Suppose we use the method used on pyramids and compare a cone with a cylinder having the same height and same size base. Take Models 18 and 8. Compare their bases. Are the circles the same size? Do the two models appear to have equal heights? How did you test this?

Now fill Model 18 with salt and empty it into Model 8. Continue until Model 8 is full. On the basis of this experiment, the volume of Model 8 is how many times that of Model 18? This illustrates the following property.

Property 7. The volume of the interior of a cone is one third that of a cylinder having the same height and whose base has the same radius.

Since we have already learned how to find the volume of a cylinder, this leads at once to the formula for finding the volume of a cone:

$$V = \frac{1}{3}\pi r^2 h.$$

Since πr^2 is B, the number of square units of area in the base, the formula could be written as

$$V = \frac{1}{3} Bh$$

Comparing this with Property 6 shows that we have the same rule for finding the volume of a cone as for a pyramid.

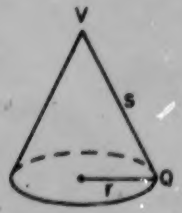
As an example, refer back to the cone mentioned above where the radius of the base was 10 inches and the height 24 inches. Then $r = 10$, $h = 24$; so by the formula above,

$$V = \frac{1}{3} \pi (10)^2 24 = 800\pi$$

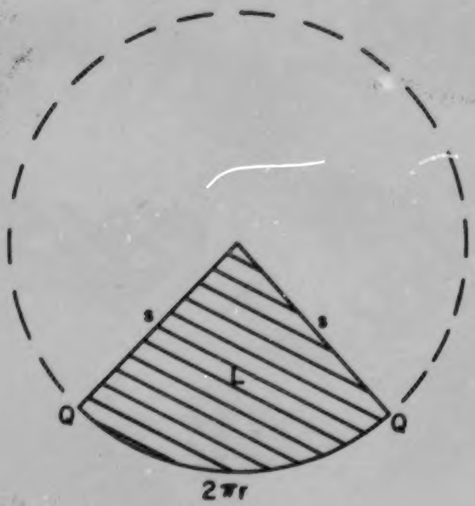
and the volume is 800π cu. in. or about 2512 cu. in.

Lateral Area of a Cone

To find the lateral area of a cone, look at Model 18. If we take it apart again, the lateral surface goes back into a sector of a circle as shown in the pattern for the model. (Notice that a sector of a circle is bounded by two radii and a part of the circle.) That is, the model which looks like this,



flattens out into a sector of a circle that looks like this.



The lateral area of the cone has the same measure as the area of the shaded part we are trying to find. The two points marked Q in the figure come from the same point of the model. The rest of the large circle is shown in dotted lines to help you follow the reasoning. Let s be the number of units in the slant height of the cone and r be the number of units in the radius of its base.

Now, in a sector of a circle, such as we have above, the area is proportional to the arc. For example, if the arc between the two points marked Q is one quarter of the circle. then the shaded region is one quarter of the interior of the circle. The circumference of the circle is $2\pi s$: its area is πs^2. If L represents the number of square units in the shaded region, we find the following proportion:

$$\frac{2\pi r}{2\pi s} = \frac{L}{\pi s^2}$$

If you multiply both sides of the equation by πs^2, what value do you find for L?

This reasoning justifies the following conclusion:

Property 8. If the slant height of a right circular cone is s units and the radius of its base r units, the number L of square units in its lateral area is given by the formula:

$$L = \pi r s.$$

As an example, refer again to the cone where the radius of the base is 10 inches long and the height 24 inches. You recall we found the slant height to be 26 inches. In this problem we have, therefore, $r = 10$; $s = 26$, so

$$L = \pi\, 10 \cdot 26 = 260\,\pi \approx 816.4$$

and the lateral area is about 816.4 square inches.

Exercises 11-7

1. If T stands for the number of square units in the total area of the cone (counting the base) write a formula for T in terms of r and s.
2. The slant height of a cone is 12 ft. and the radius of its base 3 ft. Find its lateral area and its total area in terms of π.
3. A cone has a height of 12 ft. and its slant height is 15 ft. Find the radius, the lateral area, the total area, and the volume.
4. The radius of the base of a cone is 15 inches and the volume is 2700π cubic inches. Find its height, slant height, and lateral area.
5. Let c stand for the number of units in the circumference of the base of a right circular cone. Show that the lateral area of this cone is given by the formula:

   $$L = \frac{1}{2} cs,$$

   where s stands for the slant height.

6. Look back at Problem 7 in Exercises 11-5 and show for a regular pentagonal pyramid that the lateral area is

$$L = \frac{1}{2} ps$$

where p is the number of units in the perimeter of the base and s is the altitude of each face.

7. Show why the formula in problem 6 holds for any regular pyramid.

8. Find a relationship between the formulas in problem 5 and problem 6.

9. Suppose that in the diagram for Model 18, the angle is 216° instead of $83^\circ 30'$, and that the slant height is 5" instead of $4\frac{1}{8}$". Find the lateral area of the cone. Find its volume.

10. Construct Models 19a, 19b, and 19c. Actually 19a and 19b are identical except for the lettering and can be cut out at the same time. Be sure to put the letters on, as we will need them to identify the different vertices. Notice that the letters do not refer to particular angles but identify a particular vertex after the model is assembled.

11-8. Dissection of a Prism

According to our experiments with pyramids, the volume of a pyramid is one third that of a prism having the same height as the pyramid and having a base which is congruent to the base of the pyramid. It is natural to ask whether we could see this by putting together three identical pyramids to form the prism. Unfortunately, a little experimentation seems to show this is not possible. However, we can get a kind of substitute, as we shall see.

Examine Models 19a, 19b, and 19c. They are all tetrahedrons, or triangular prisms. First compare Models 19a and 19b. How does face ABC of Model 19a compare with face SRQ of Model 19b? How do their heights compare if we consider these faces as bases?

(Actually these questions are a little ridiculous since we have already noticed the patterns are identical for the two models so all their measurements must agree.) In any case the two tetrahedrons ABCQ and QRSC (that is, Models 19a and 19b) have interiors with equal volumes.

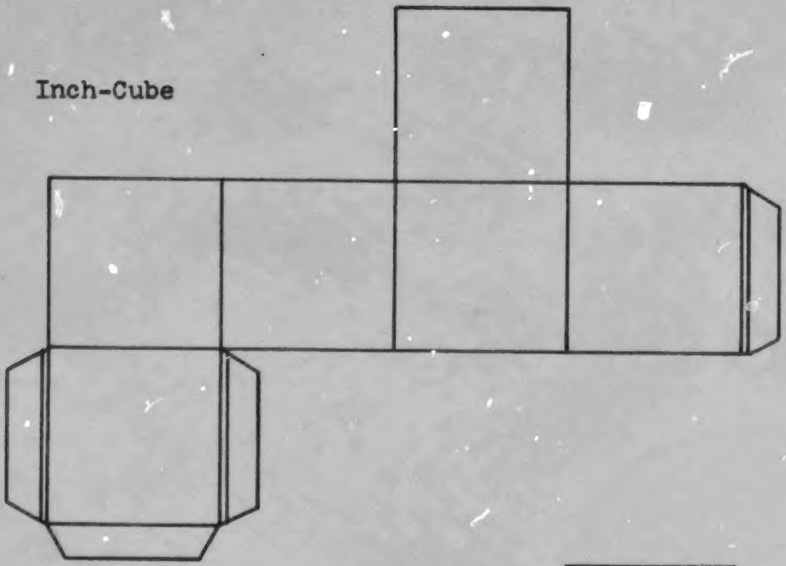
Now compare Models 19a and 19c. We find these models definitely do not look alike. However, compare face ABQ of Model 19a with face BCR of 19c. Do you find them congruent? Place the models on the desk with these faces in contact with the top of the desk. Notice that in these positions you can push the models together so that the two faces marked BCQ coincide. What can you say of the heights of these two models when placed in this position? Models 19a and 19c, when looked at in this way, are triangular pyramids with congruent bases and congruent altitudes. What can you say about their volumes? What property are you using?

You should have concluded that the three Models 19a, 19b, 19c have equal volumes. Now put the three models together so that faces BCQ of Models 19a and 19c coincide and so that faces QRC of 19b and 19c coincide. What is the resulting figure? Is it a triangular prism?

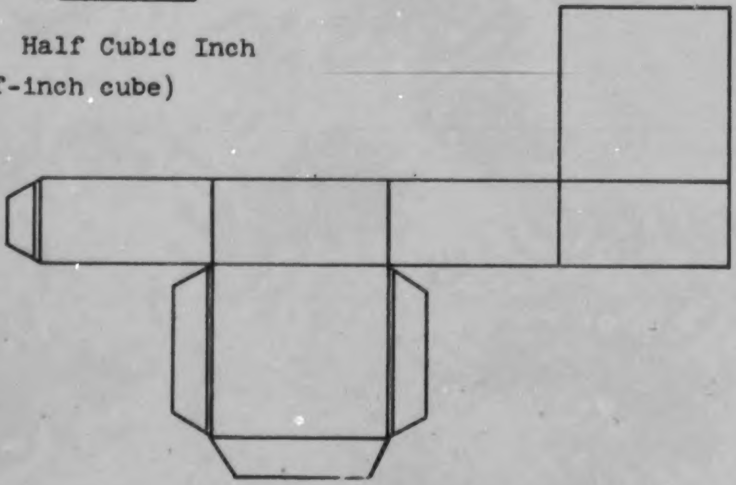
These three models with equal volumes can thus be assembled to form a prism whose base is the same as the face ABC of Model 19a, and whose height is the same as that of 19a. This shows again the result stated in Property 6. Actually the work is just discussed in Chapter 10 except that here we have been particularly interested in the volumes of the pieces.

If we imagine Model 19a as originally given, we can think of Models 19b and 19c as two more tetrahedrons which have been invented having the same volume as 19a, and so that they can be combined with 19a to produce a prism of the same base and height. In this particular case the base of 19a is an equilateral triangle, and one of the lateral edges is perpendicular to the plane of the base. Could this still have been done if ABCQ were any triangular prism? The answer is, yes.

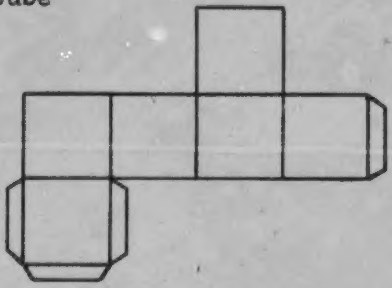
Model 1. Inch-Cube



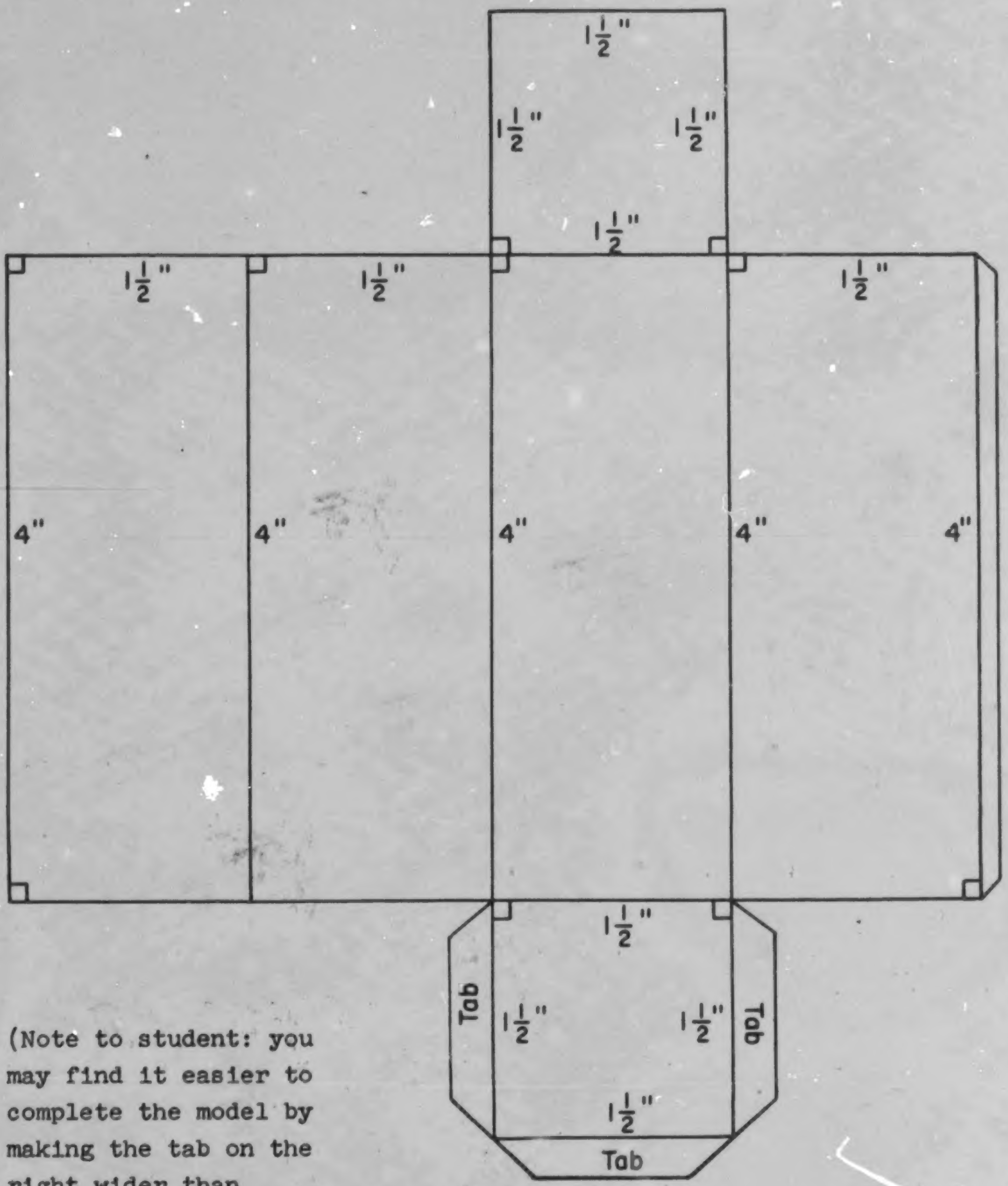
Model 2. Half Cubic Inch
(not half-inch cube)



Model 3. Half-inch Cube



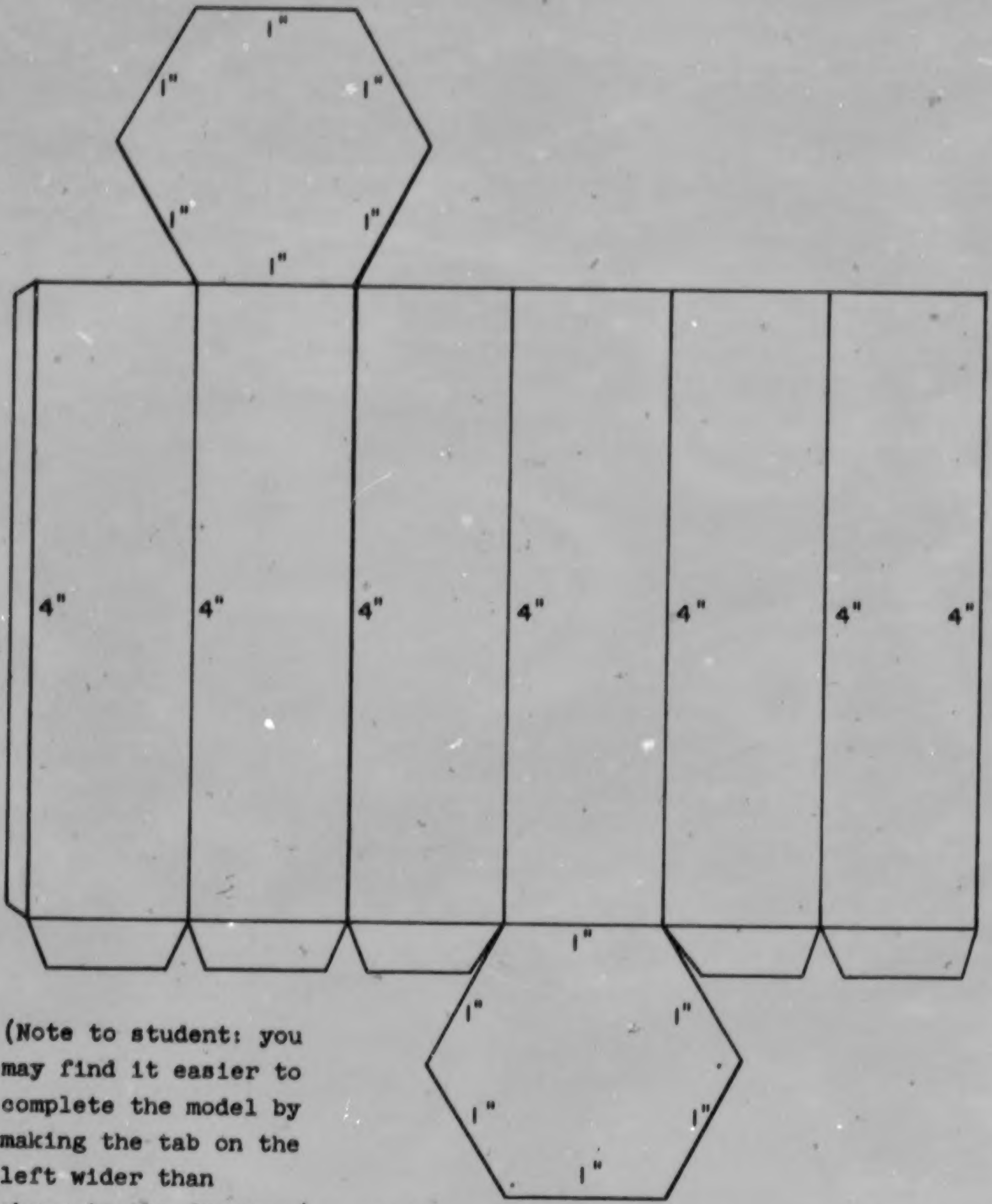
Model 4. Rectangular Right Prism


$1\frac{1}{2}$"
$1\frac{1}{2}$"
$1\frac{1}{2}$"
$1\frac{1}{2}$"
$1\frac{1}{2}$"
$1\frac{1}{2}$"
$1\frac{1}{2}$"
4"
4"
4"
4"
4"
$1\frac{1}{2}$"
Tab
$1\frac{1}{2}$"
$1\frac{1}{2}$"
Tab
$1\frac{1}{2}$"
Tab

(Note to student: you may find it easier to complete the model by making the tab on the right wider than shown in the drawing.)

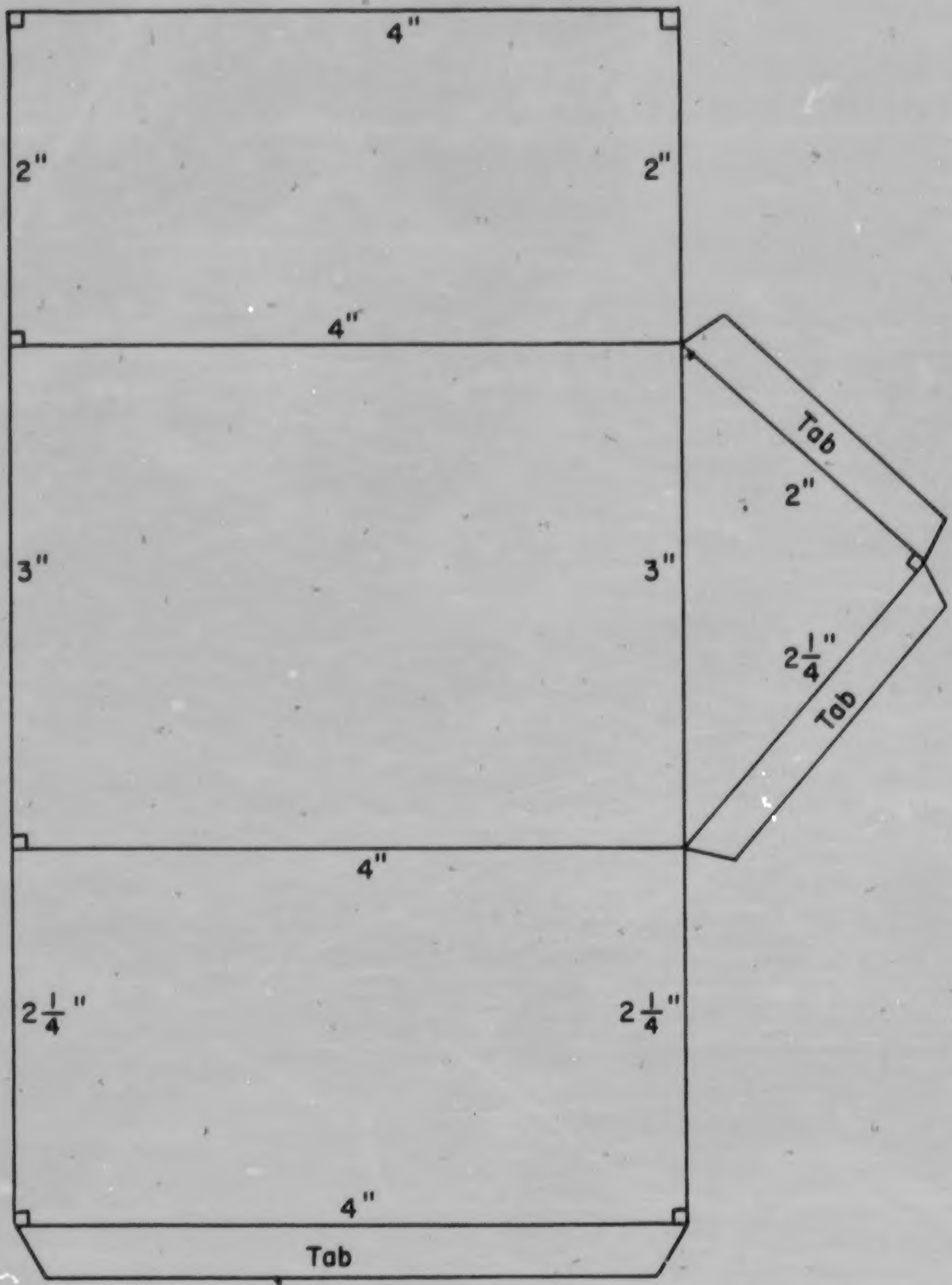
[sec. 11-8]

Model 5. Right Hexagonal Prism

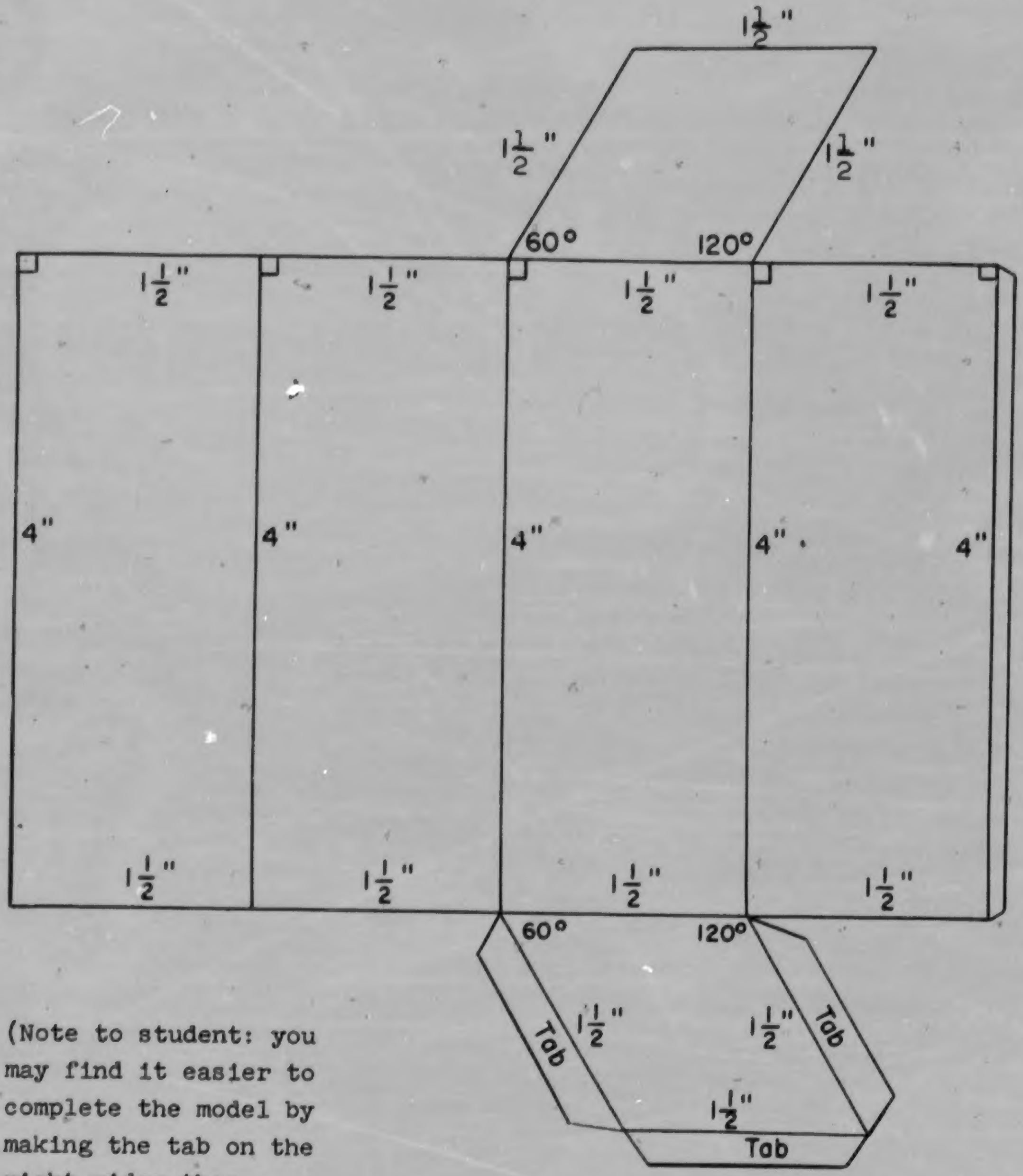

1"
1"
1"
1"
1"
1"
4"
4"
4"
4"
4"
4"
4"
1"
1"
1"
1"
1"
1"

(Note to student: you may find it easier to complete the model by making the tab on the left wider than shown in the drawing.)

Model 6. Right Triangular Prism (Make an extra copy of the triangle for the top base. Use only one tab so the top can be opened.)

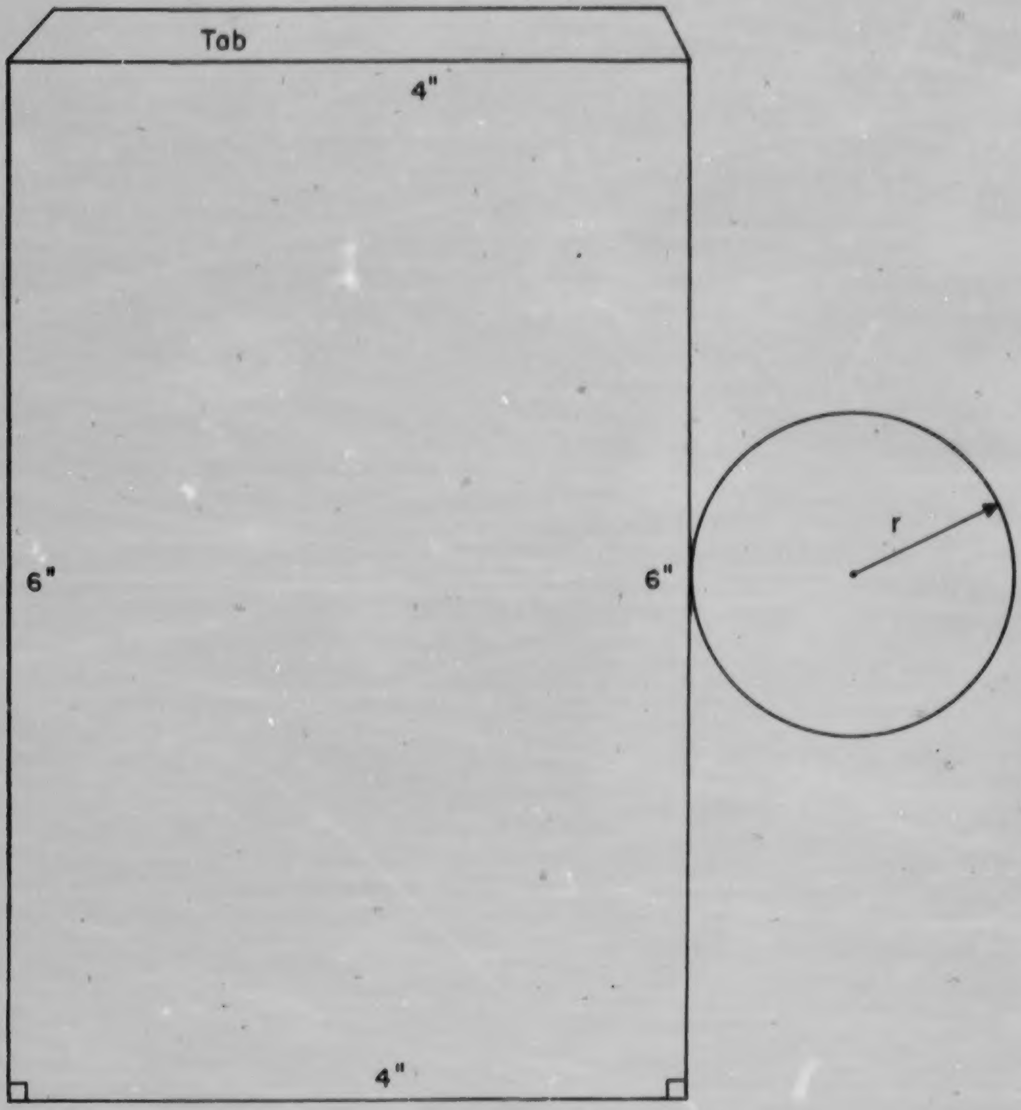

4"
2"
2"
4"
3"
3"
Tab
2"
$2\frac{1}{4}$"
Tab
4"
$2\frac{1}{4}$"
$2\frac{1}{4}$"
4"
Tab

Model 7. Right Prism with Rhombus as Base (also, Parallelepiped)

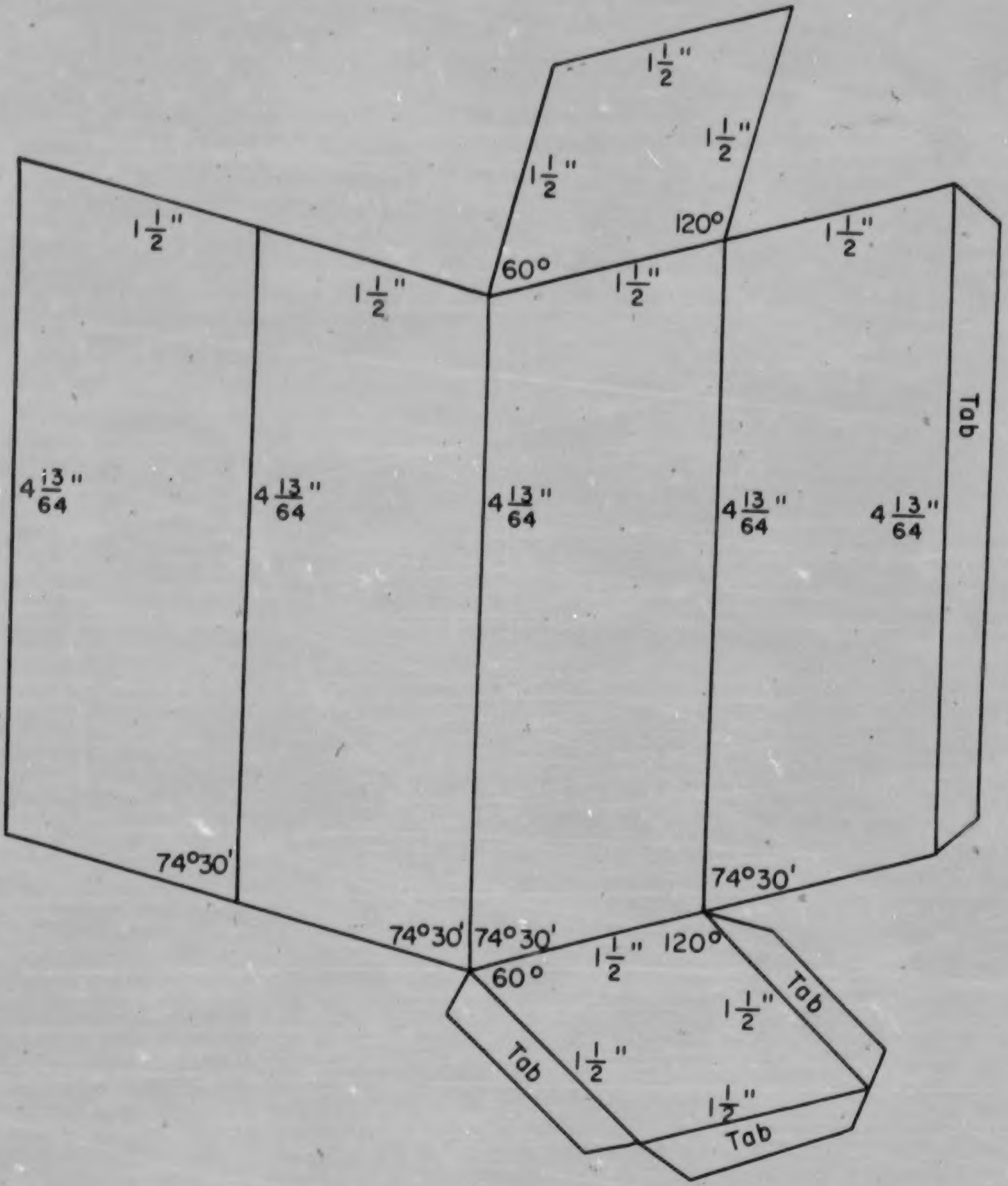

$1\frac{1}{2}$"
$1\frac{1}{2}$"
$1\frac{1}{2}$"
60°
120°
$1\frac{1}{2}$"
$1\frac{1}{2}$"
$1\frac{1}{2}$"
$1\frac{1}{2}$"
4"
4"
4"
4"
4"
$1\frac{1}{2}$"
$1\frac{1}{2}$"
$1\frac{1}{2}$"
$1\frac{1}{2}$"
60°
120°
Tab
$1\frac{1}{2}$"
$1\frac{1}{2}$"
Tab
$1\frac{1}{2}$"
Tab

(Note to student: you
may find it easier to
complete the model by
making the tab on the
right wider than
shown in the drawing.)

Model 8. Right Circular Cylinder (Draw the circular bases with your own compass, using the radius of the circle below. Make an extra copy of the circle, since there are two bases. Attach the lower base firmly (with tape) but attach the top base only at one point so it can be readily opened.)

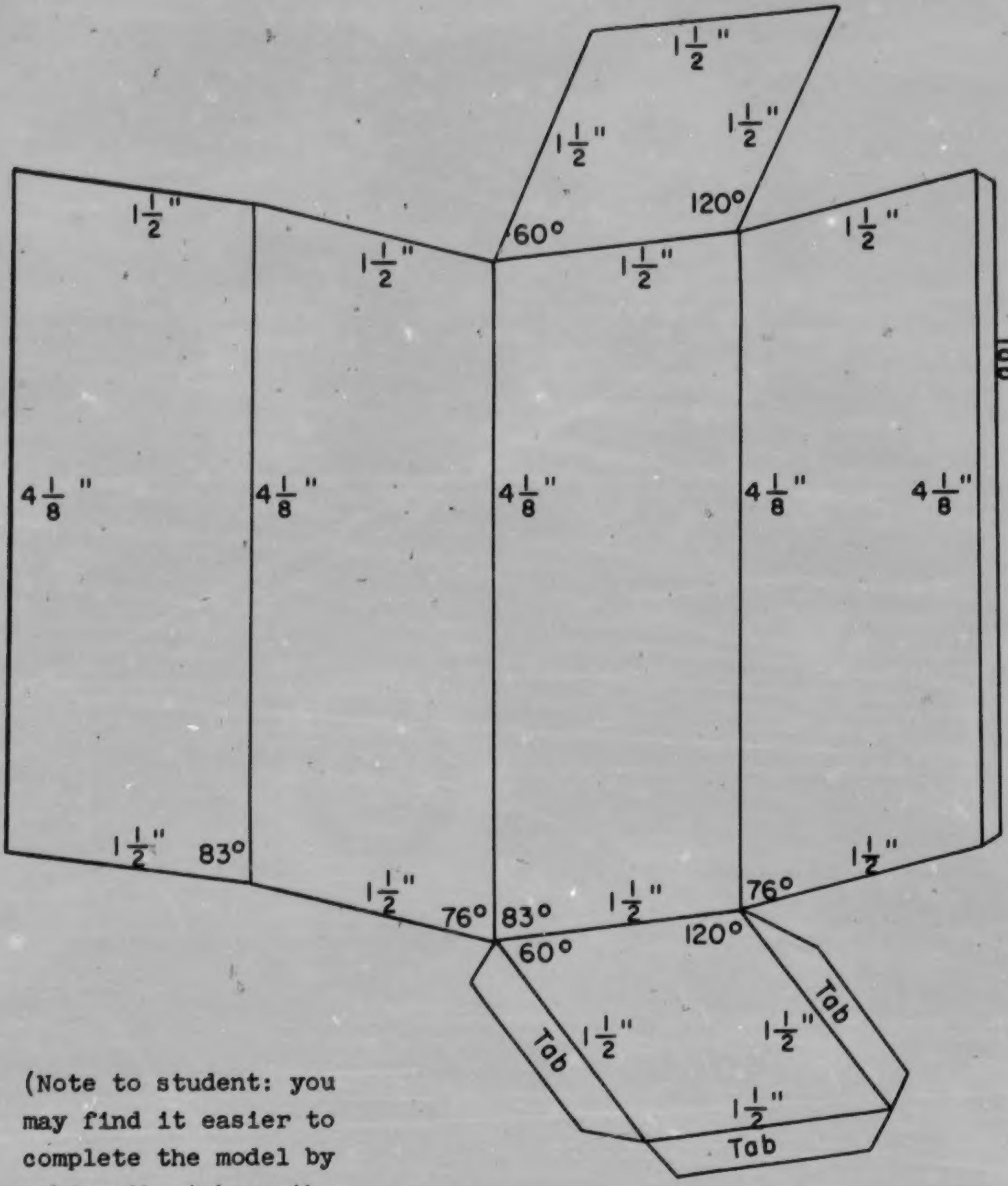


Model 9. Oblique Prism with Rhombus as Base (also, Parallelepiped)


$1\frac{1}{2}$"
$1\frac{1}{2}$"
$1\frac{1}{2}$"
120°
60°
$1\frac{1}{2}$"
$1\frac{1}{2}$"
$1\frac{1}{2}$"
$1\frac{1}{2}$"
Tab
$4\frac{13}{64}$"
$4\frac{13}{64}$"
$4\frac{13}{64}$"
$4\frac{13}{64}$"
$4\frac{13}{64}$"
74°30'
74°30'
74°30'
74°30'
$1\frac{1}{2}$"
120°
60°
Tab
$1\frac{1}{2}$"
Tab
$1\frac{1}{2}$"
$1\frac{1}{2}$"
Tab

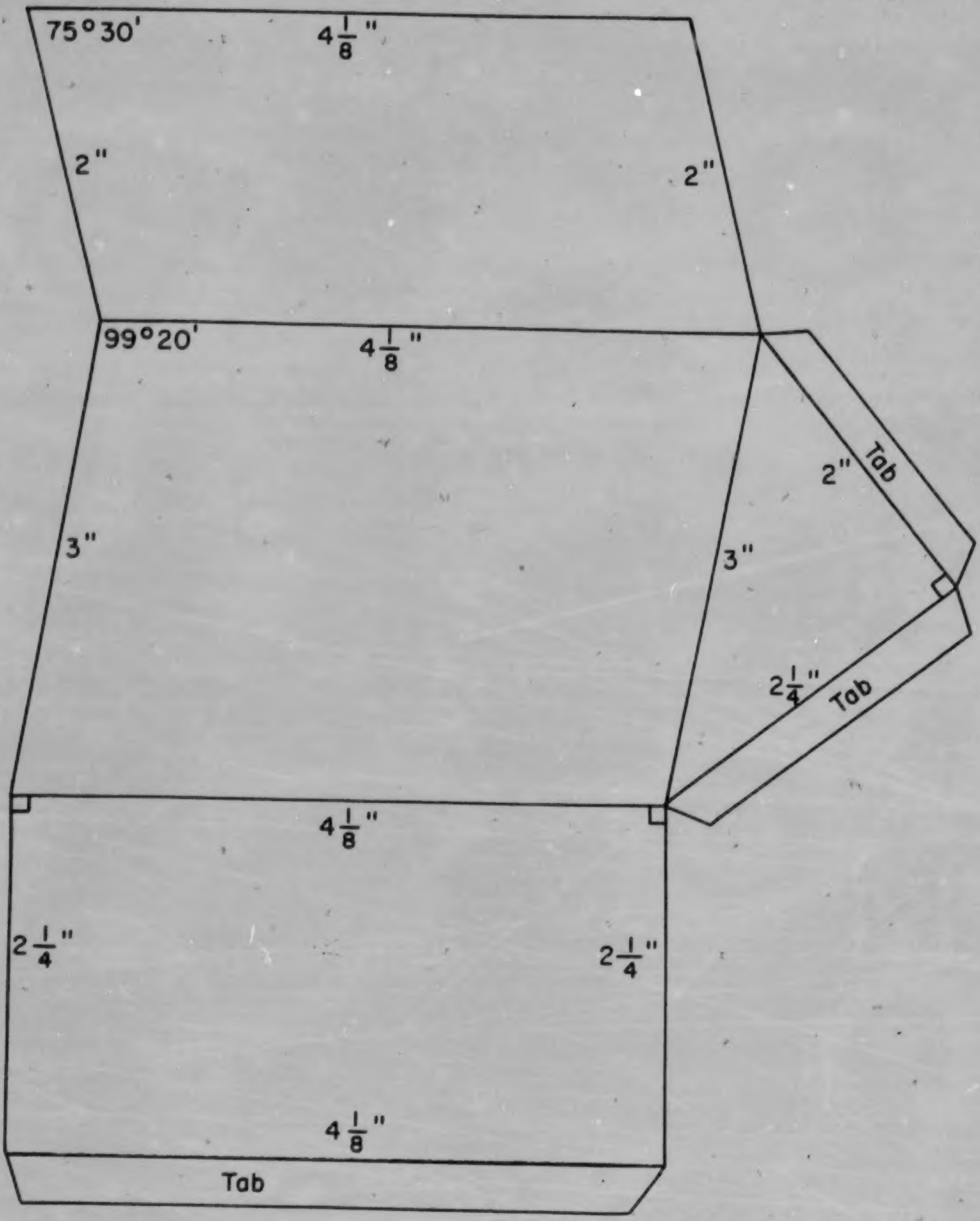
[sec. 11-8]

Model 10. Oblique Prism with Rhombus as Base (also, Parallelepiped)


1 1/2"
1 1/2"
1 1/2"
120°
60°
1 1/2"
1 1/2"
1 1/2"
1 1/2"
Tab
4 1/8"
4 1/8"
4 1/8"
4 1/8"
4 1/8"
1 1/2"
83°
1 1/2"
1 1/2"
76°
83°
1 1/2"
76°
1 1/2"
120°
60°
Tab
1 1/2"
1 1/2"
Tab
1 1/2"
Tab

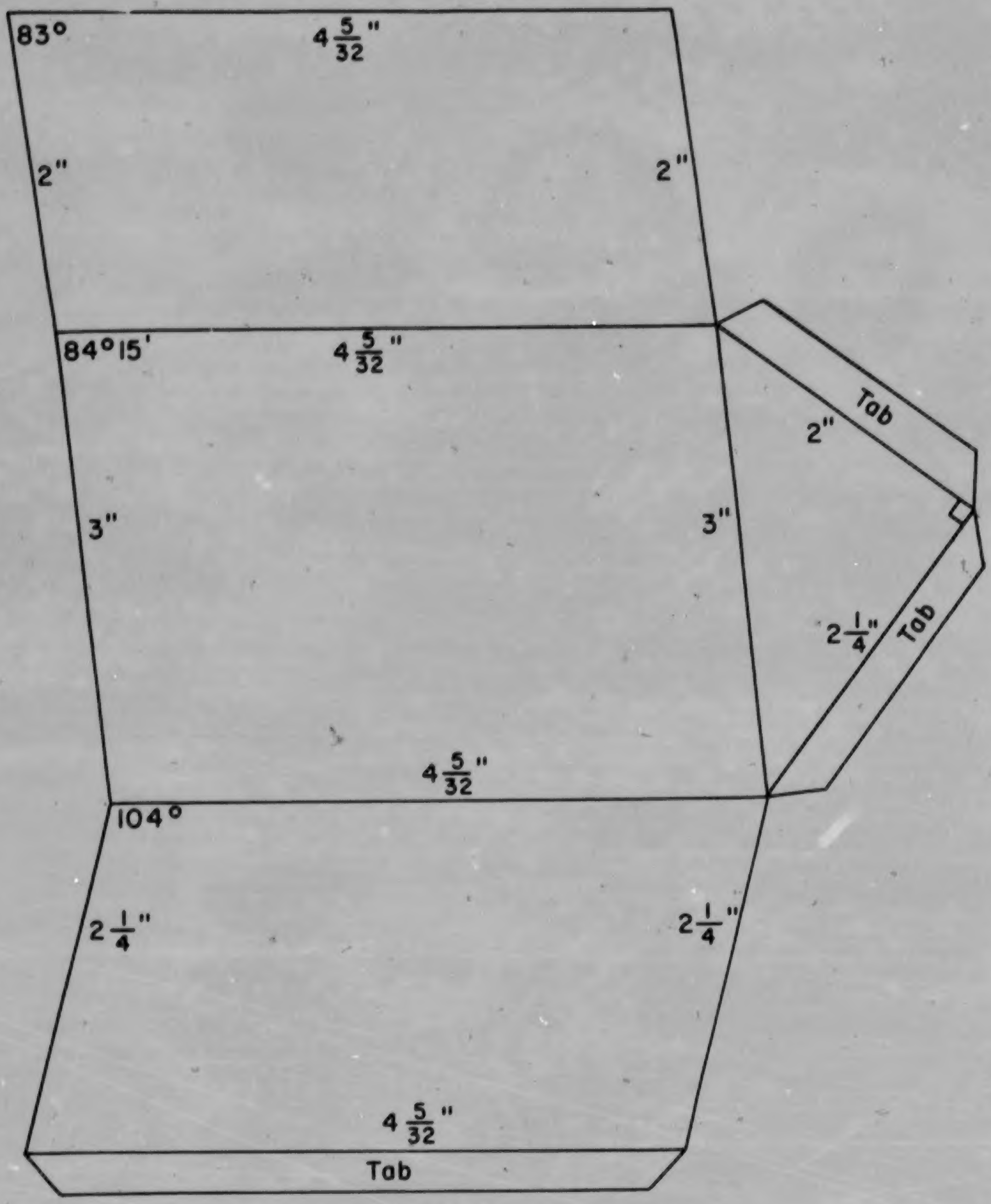
(Note to student: you
may find it easier to
complete the model by
making the tab on the
right wider than
shown in the drawing.)

Model 11. Oblique Triangular Prism (Make an extra copy of the triangle for the other base. Use only one tab in attaching it so the top can be opened if desired.)

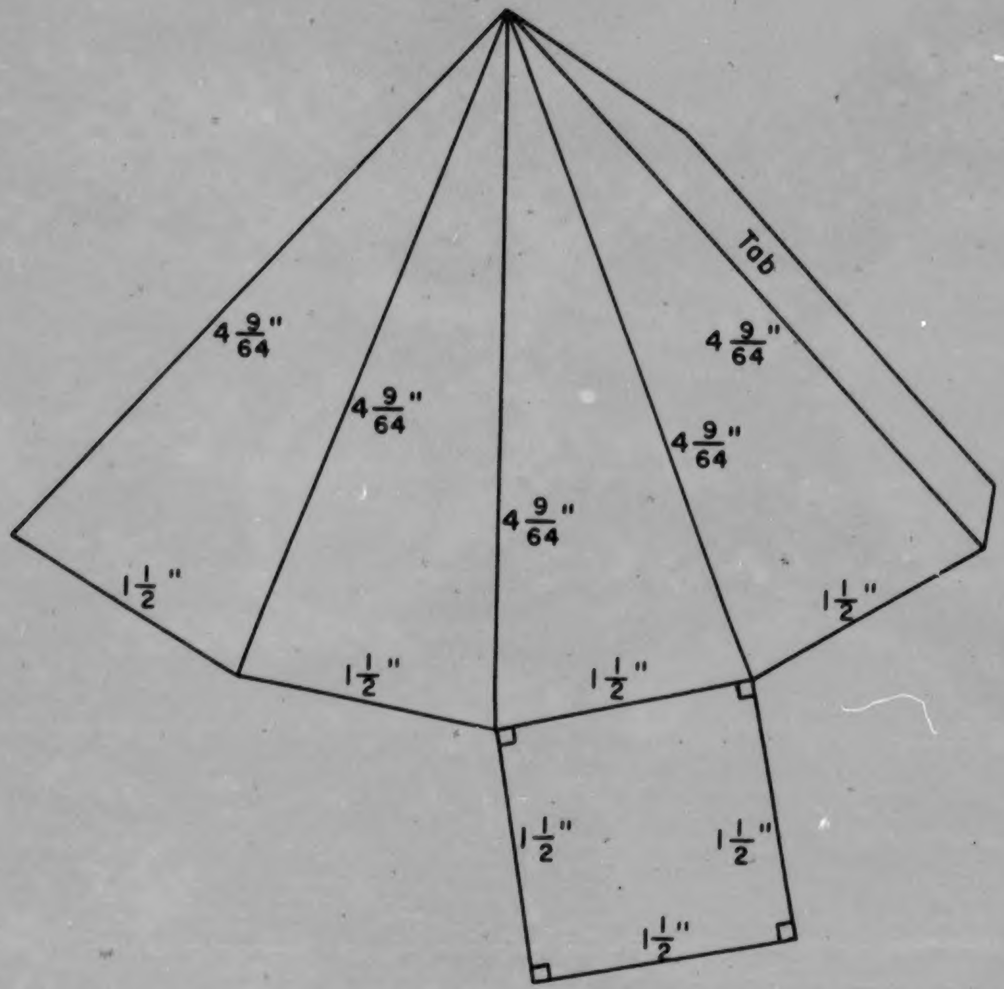

75°30'
4 1/8"
2"
2"
99°20'
4 1/8"
3"
3"
2"
Tab
2 1/4"
Tab
4 1/8"
2 1/4"
2 1/4"
4 1/8"
Tab

[sec. 11-8]

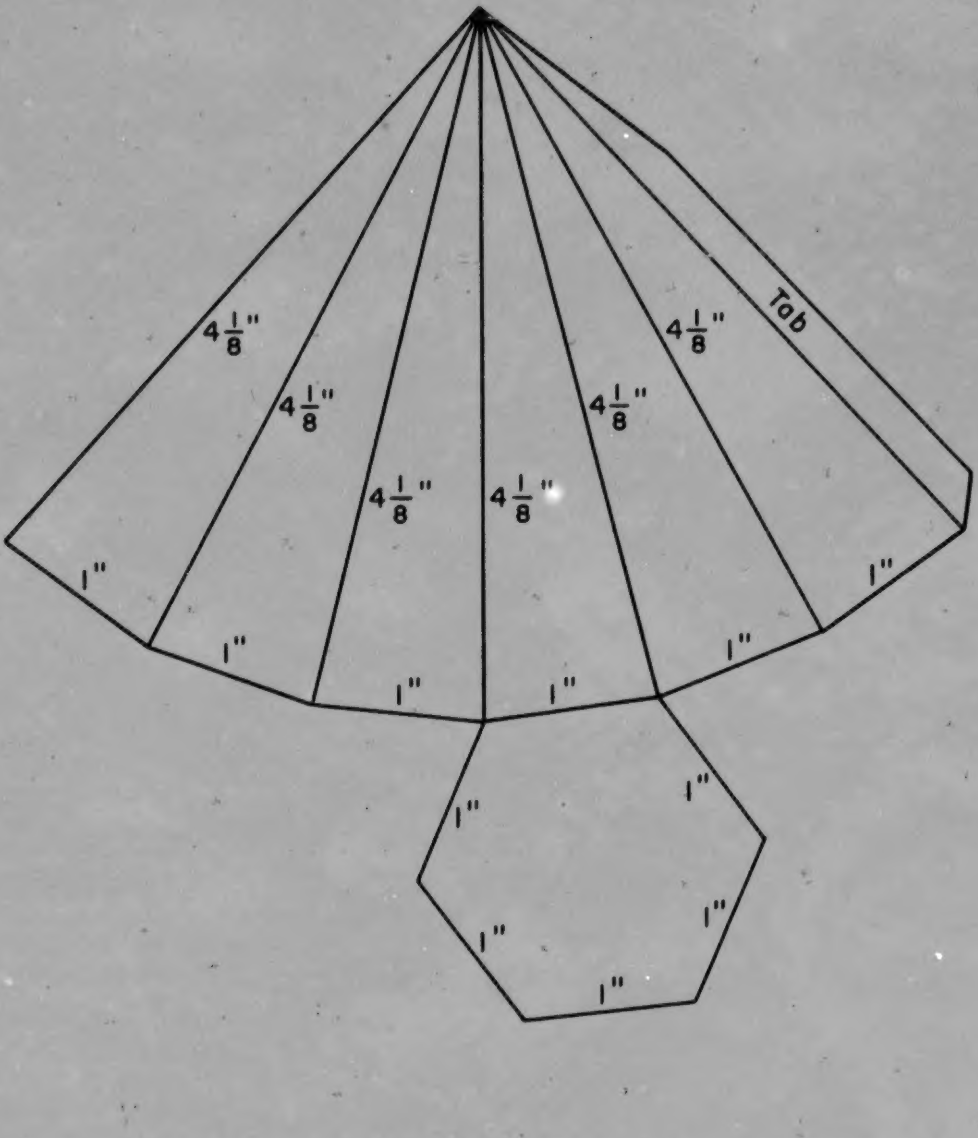
Model 12. Oblique Triangular Prism (Make an extra copy of the triangle to use for the other base. Use only one tab in attaching it so the top can be opened if desired.)


83°
4 5/32"
2"
2"
84°15'
4 5/32"
3"
3"
Tab
2"
2 1/4"
Tab
4 5/32"
104°
2 1/4"
2 1/4"
4 5/32"
Tab

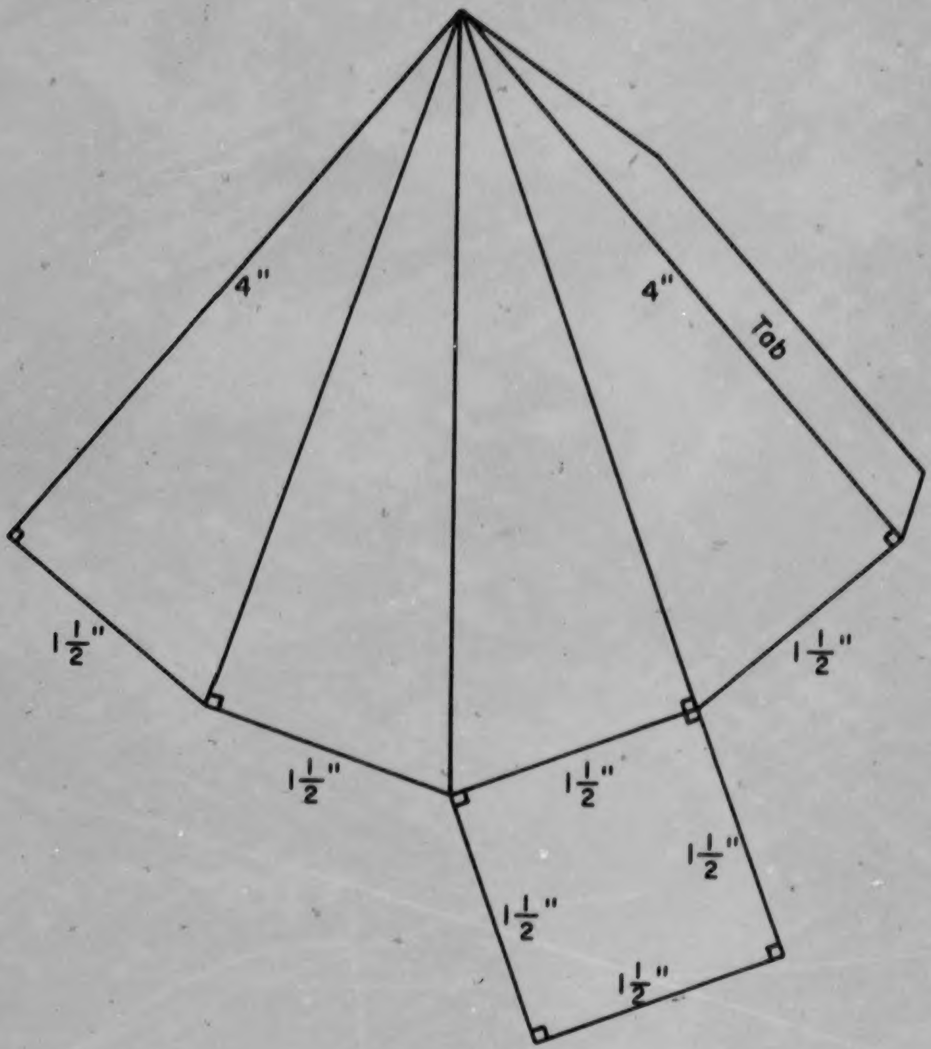
Model 13. Regular Square Pyramid


Tab
4 9/64"
4 9/64"
4 9/64"
4 9/64"
4 9/64"
1 1/2"
1 1/2"
1 1/2"
1 1/2"
1 1/2"
1 1/2"
1 1/2"

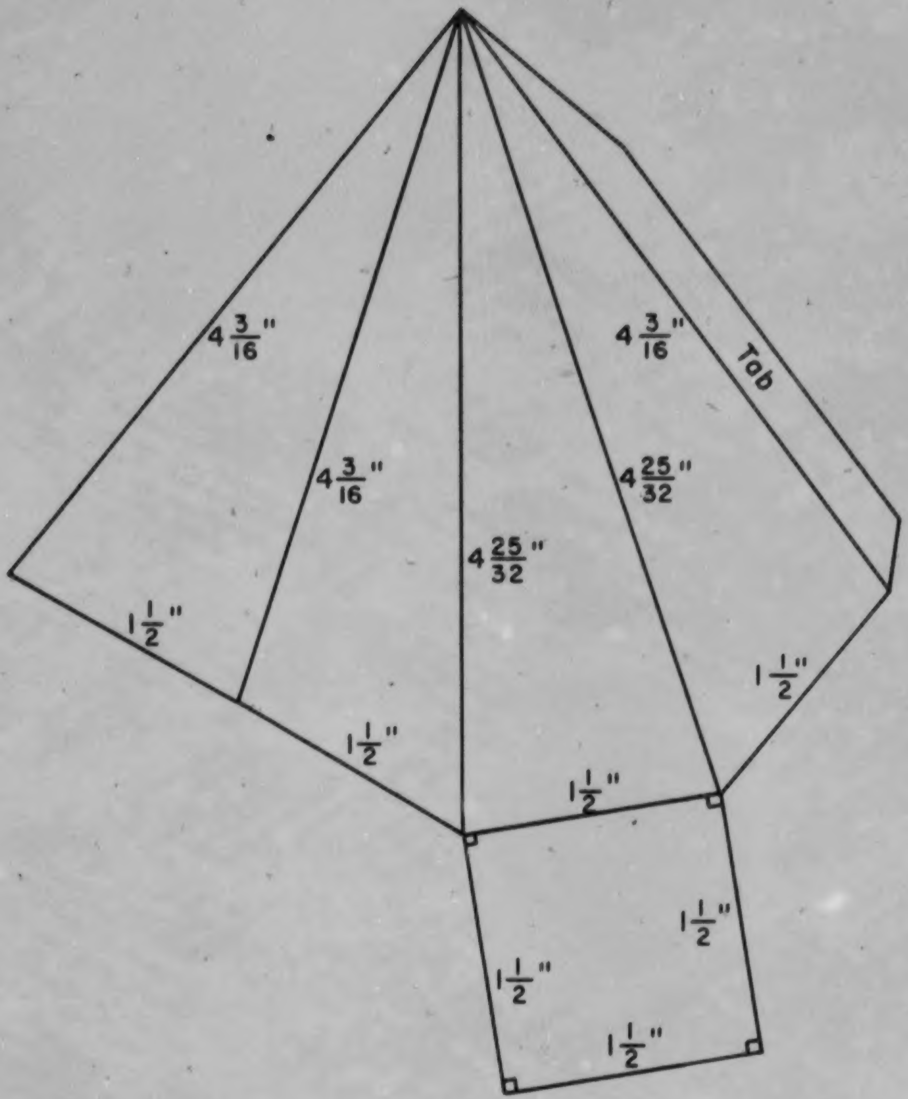
Model 14. Regular Hexagonal Pyramid


4 1/8"
4 1/8"
4 1/8"
4 1/8"
4 1/8"
4 1/8"
Tab
1"
1"
1"
1"
1"
1"
1"
1"
1"
1"
1"
1"

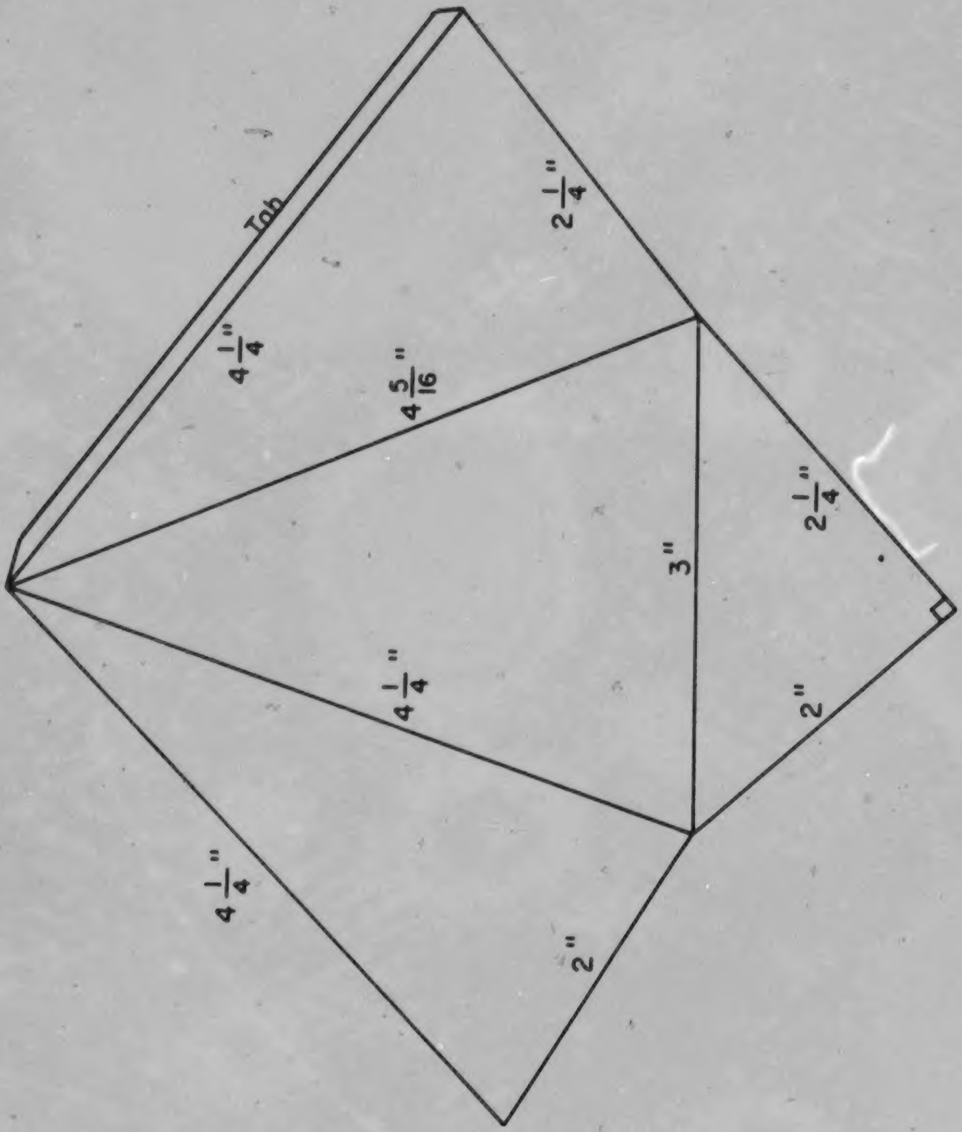
Model 15. Square Pyramid


4"
4"
Tab
$1\frac{1}{2}$"
$1\frac{1}{2}$"
$1\frac{1}{2}$"
$1\frac{1}{2}$"
$1\frac{1}{2}$"
$1\frac{1}{2}$"
$1\frac{1}{2}$"

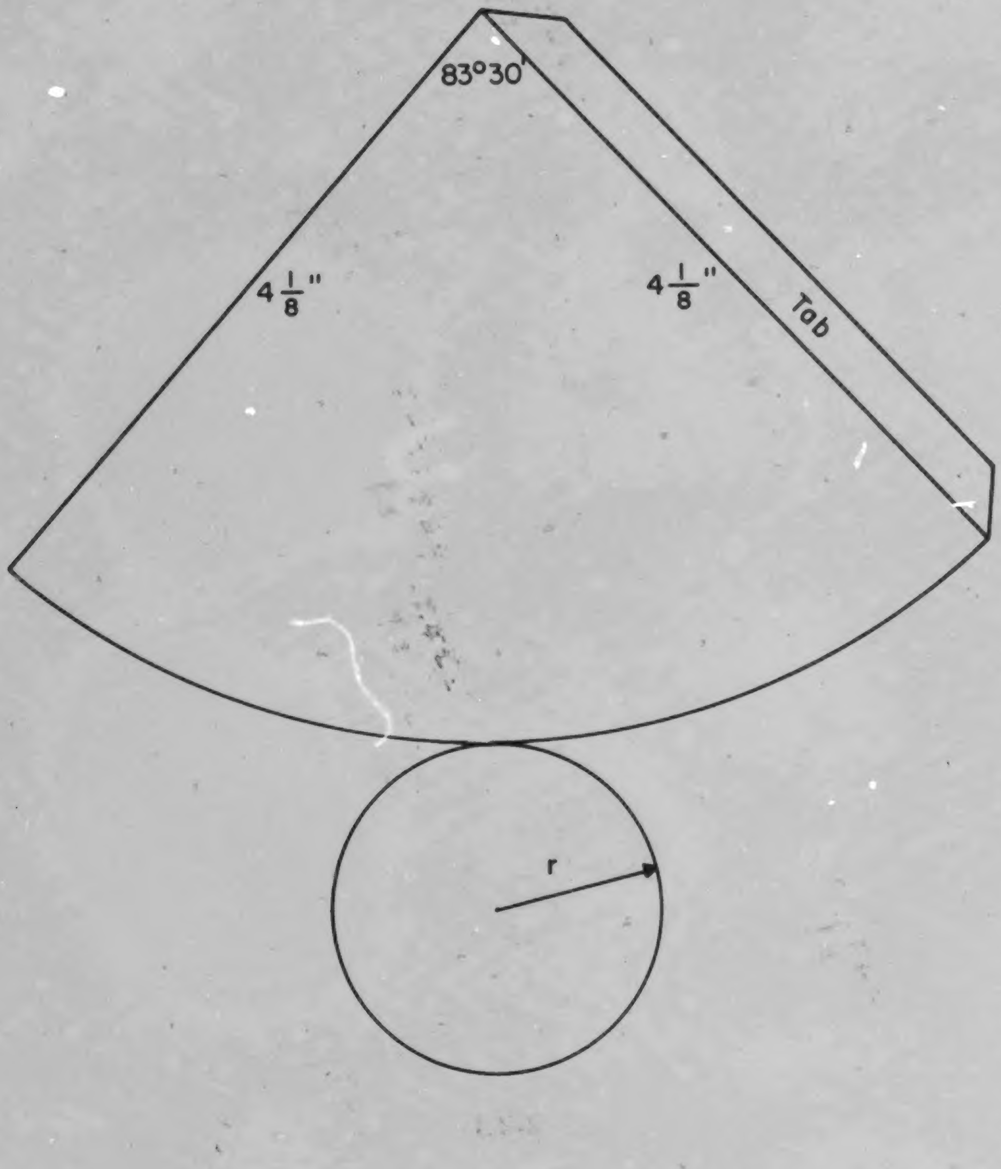
Model 16. Square Pyramid


$4\frac{3}{16}"$
$4\frac{3}{16}"$
$4\frac{3}{16}"$
Tab
$4\frac{25}{32}"$
$4\frac{25}{32}"$
$1\frac{1}{2}"$
$1\frac{1}{2}"$
$1\frac{1}{2}"$
$1\frac{1}{2}"$
$1\frac{1}{2}"$
$1\frac{1}{2}"$
$1\frac{1}{2}"$

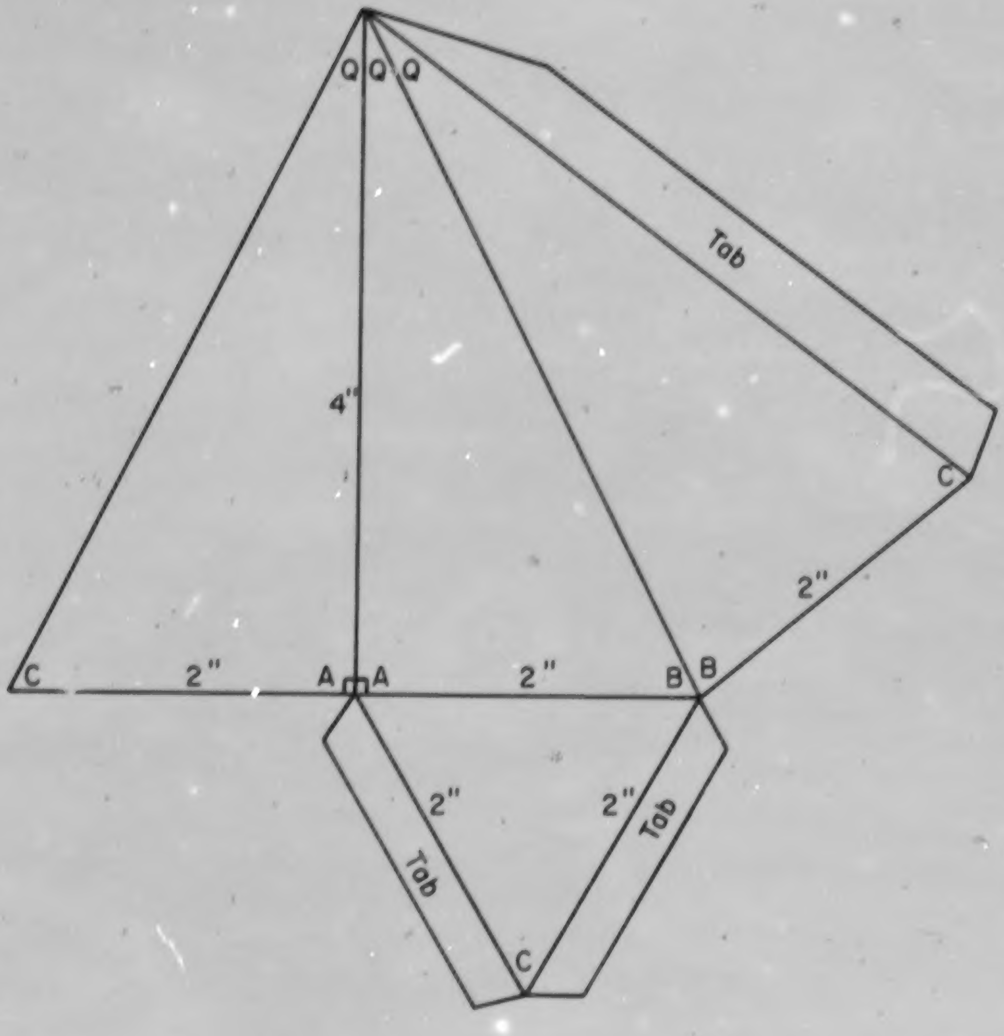
Model 17. Triangular Pyramid (Tetrahedron)


Tab
$2\frac{1}{4}''$
$4\frac{1}{4}''$
$4\frac{5}{16}''$
$2\frac{1}{4}''$
$3''$
$4\frac{1}{4}''$
$2''$
$4\frac{1}{4}''$
$2''$

Model 18. Right Circular Cone (Draw the circle and arc with your own compass using the radii shown. The radius of the small circle is suppose to be the same as in Model 8.)

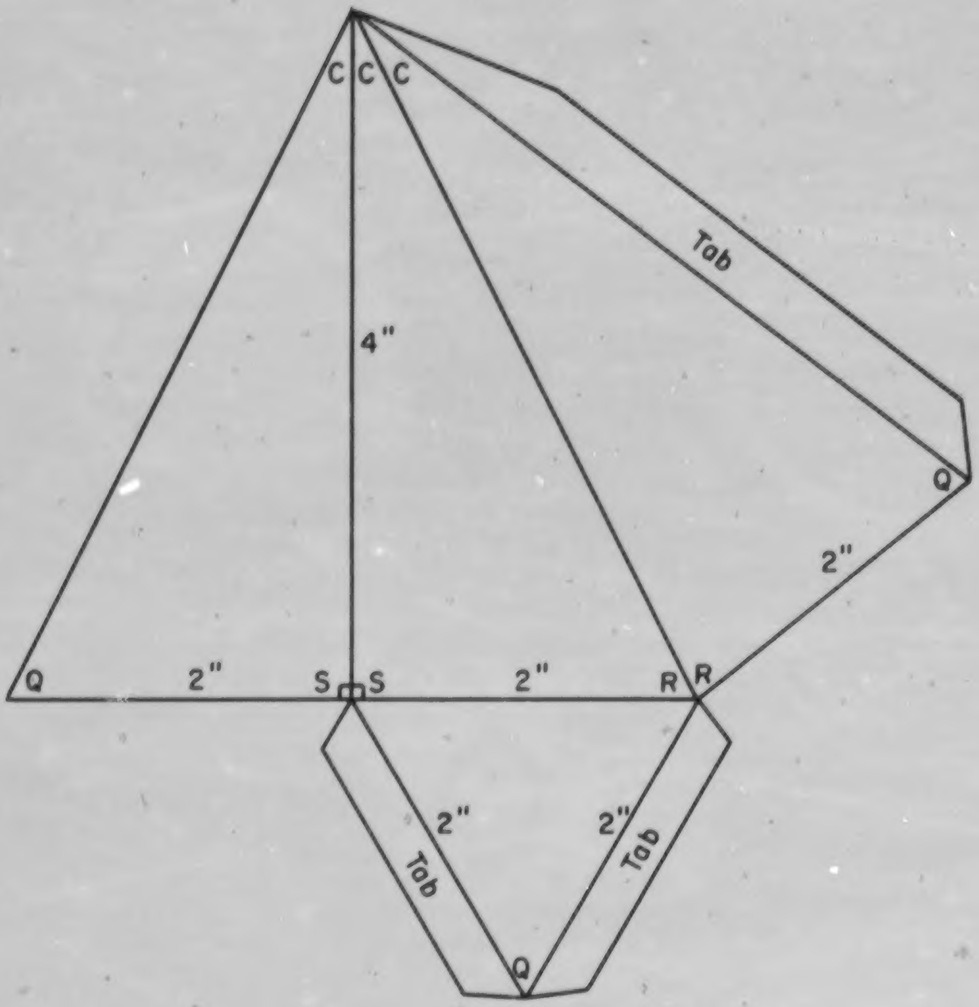


Model 19a

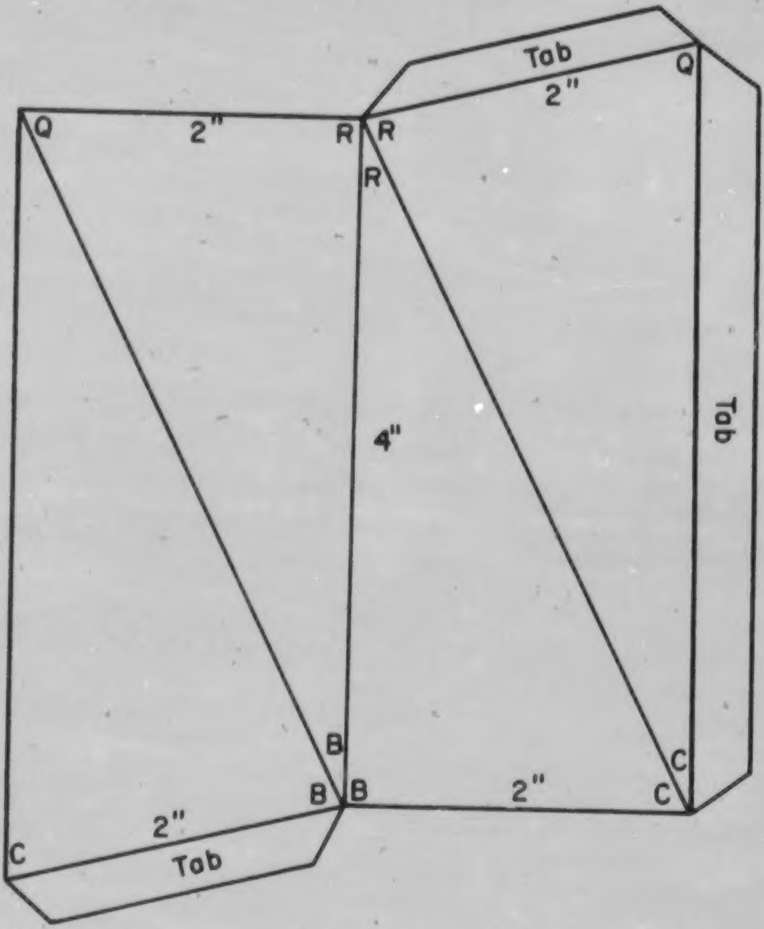

Q
Q
Q
Tab
4"
C
2"
C
2"
A
A
2"
B
B
2"
2"
Tab
Tab
C

[sec. 1148]

Model 19b


C
C
C
4"
Tab
Q
2"
Q
2"
S
S
2"
R
R
2"
2"
Tab
Tab
Q

Model 19c


Tab
Q
2"
Q
2"
R
R
R
4"
Tab
B
B
B
2"
C
C
2"
C
Tab

The remaining segments not labeled for length are the same lengths as the segments in Models 19a and 19b joining the same endpoints. That is, segment BQ here has the same length as segment BQ in Model 19a.

Summary of Properties given in Chapter 11

Property 1. If a line is perpendicular to two distinct intersecting lines in a plane, it is perpendicular to the plane.

Property 2. The shortest segment from a point Q outside a plane r to the plane r is the segment perpendicular to that plane.

Property 3. If two planes are parallel, the perpendicular distances from different points of one plane to the other plane are all the same.

Property 4. If two prisms have congruent bases and equal heights they have equal volumes.

Property 5. If two pyramids have congruent bases and congruent altitudes, they have equal volumes.

Property 6. The volume of a pyramid is one third that of a prism whose base is congruent to the base of the prism and whose height is the same as that of the prism. ($V = \frac{1}{3} Bh$, where B stands for the number of square units of area in the base and h the number of linear units in the height.)

Property 7. The volume of the interior of a cone is one third that of a cylinder having the same height and whose base has the same radius. ($V = \frac{1}{3} \pi r^2 h$ or $V = \frac{1}{3} Bh$)

Property 8. If the slant height of a right circular cone is s units, and the radius of its base r units, the number L of square units in its lateral area is given by the formula,

$$L = \pi rs.$$

Chapter 12

THE SPHERE

12-1. Introduction

If you were asked to describe the shape of a coin, you might say that it is "round." But this definition is vague. To describe it more accurately, you might say that it is "circular." This is more accurate because you have already learned a careful definition of a circle:

> A circle is a set of points in a plane such that all points of the set lie at the same distance from a particular point, P, called the center.

Recall how you use a compass to draw a circle with a given center and radius. The point of the compass is placed at the center. The compass holds the point of the pencil at a constant distance (the radius) from the center as you draw the circle. We call the drawing a circle, but the drawing merely represents the circle, just as a drawing of a line segment represents a line segment.

When we talk about circles, we talk about points in a plane. Suppose we consider all the points in space. What subset of points is suggested by the following description: "A set of points in space, each point of the set at the same distance from a particular point"? This set of points would be more than a circle. We would have a surface, like the surface of a ball. Such a surface we call a sphere. The point from which the distances are measured is called the center of the sphere. Some people might call such a surface "perfectly round" but this is not as definite a term as "sphere."

The surface of the earth is a fairly good representation of a sphere. But it is not exactly a sphere because of its mountains and its valleys. Also, the earth is somewhat flattened at the poles. (The length of the equator is 24,902 miles and that of a great circle through the poles is 24,860 miles--like most mature bodies, it is slightly larger around the middle!) The surface of a basketball is a better representation of a sphere. The surface of some Christmas tree ornaments, or the surface of a BB shot, are even better representations of a sphere because they are smoother.

Many objects are spherical, that is, have the shape of a sphere. Some of these objects, such as ball bearings, are important to industry. Some, like rubber balls, are used as toys. It is because of these many spherical objects and, most of all, because of the shape of the earth, that it is important to know some of the properties of spheres.

Before we go any further we should emphasize that it is the surface that we call the sphere. On a ball, only that portion that we could paint represents the surface. In a gas-filled weather balloon, only the portion of the balloon exposed to the air is the surface.

A very interesting representation of a sphere is a soap bubble. Have you ever wondered why a soap bubble takes the shape of a sphere? There is a very definite physical reason for this. We will learn more about this property at the end of this chapter.

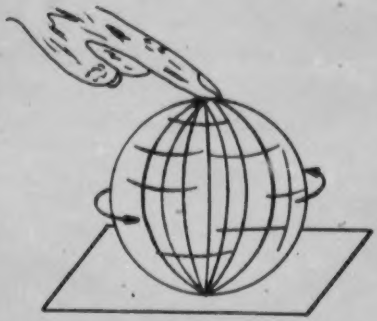
Exercises 12-1

1. List as many games as you can in which a spherical object, such as a ball, is involved.
2. List as many spherical objects as you can which are used as containers.
3. Fishing is a popular sport and an important industry. Can you think of any spherical objects which are very useful in certain kinds of fishing?
4. List some objects, useful in a home, that are shaped like spheres.

5. Consider a coin such as a fifty-cent piece. Assume the coin has no thickness.

   (a) What geometric figure does the edge of the coin suggest?

   (b) Suppose you use one finger to hold the coin in an upright position as shown at the right. When hit sharply on one edge the coin rotates very rapidly. What geometric idea is represented by the edge of the rotating coin?



   (c) Does the geometric idea represented by the rotating edge have any thickness?

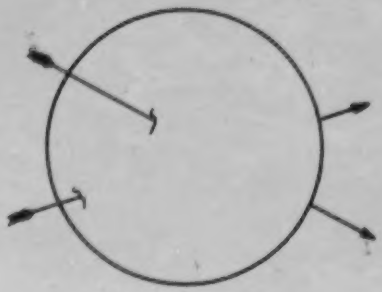
6. In the study of geometry of a plane we sometimes refer to an object as having only two dimensions. Name an object which has three dimensions.

7. Refer to the text, and review the definition of circle. Write a definition of sphere.

8. (a) Suppose all the points of a sphere are a distance v from the center, C, of the sphere. How can you describe the set of all points which are located at distances less than v from C?

   (b) How can you describe the set of points which are located at distances greater than v from C?

12-2. Great and Small Circles



Suppose we let a ball represent a sphere. Arrows, shot at the ball, pierce the surface, or skin, as shown at the right. It would be possible to shoot many such arrows, piercing and passing through the ball. Assuming that each arrow pierces the surface of the ball, as shown in the drawing, in how many distinct points must each arrow pierce the surface if the length of the arrow is greater than the diameter of the sphere (see below for definition of "diameter")?

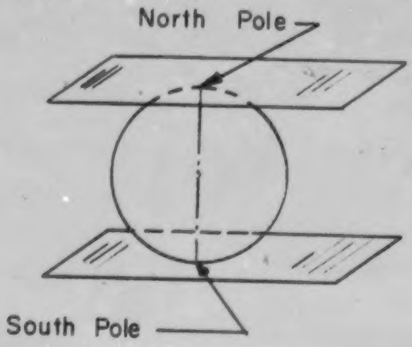
Each arrow that pierces the ball may be thought of as a line which intersects the sphere. Let us think about all such lines which intersect a particular sphere in two distinct points. Some lines will pass through the upper part of the sphere, some through the lower part, and some through both parts. We can imagine infinitely many such lines passing through the sphere. Every line which passes through the sphere intersects the sphere in two distinct points.

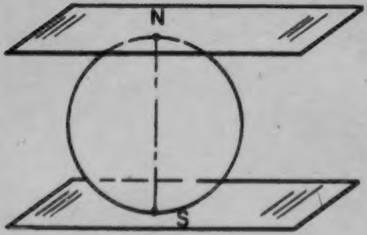
Each of these lines contains a line segment whose endpoints lie on the sphere. Let us consider this set of line segments. Are all such segments congruent, that is, do they have the same length? No, but there is one subset of these segments which are congruent--namely those segments which pass through the center of the sphere. A line segment with both endpoints on the sphere and passing through the center of the sphere is called a diameter of the sphere. Will each line segment whose endpoints are on the sphere have the same length as a diameter? Do you see why it is necessary to include "passing through the center" in our definition of a diameter?

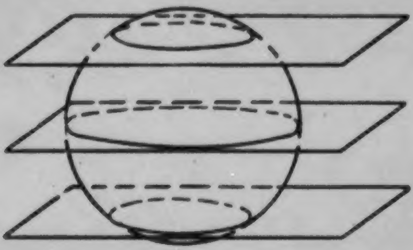
The line passing through the poles of the earth is called the axis of the earth. This is approximately the line about which the earth revolves. If we think of the earth as a sphere, the diameter contained in the axis intersects the sphere at the North Pole and at the South Pole. We think of these two points, represented by the poles, as being "directly opposite" each other. Since the prefix "anti" means opposing, we could call these points "anti-polar" points. But the poles are not the only points on earth having this property, since each diameter of the earth will contain two such points. So, we use a different name. The endpoints of any diameter of a sphere are called antipodal points. (This is pronounced "an - tip'- o - dal.") We say, then, that each endpoint of a diameter of a sphere is an antipode of the other endpoint. (In "antipode" the accent is on the first syllable.) Thus, the North Pole represents a point which is an antipode of the point represented by the South Pole. Every point on a sphere has one antipode. To find the antipode of any point P on a sphere, connect P to the center C of the sphere. The line through these two points will intersect the sphere in the antipode of point P.

As an example of antipodal points, think of the point on the surface of the earth on which you are standing. The antipode of this point would be on the far side of the earth. You might think of a hole dug straight down through the center of the earth, coming out on the other side. Where would it come out? You might find it interesting to locate the antipode of the place where you live. To do so, use a globe representing the earth.

Now think of the earth as a sphere with a vertical axis as shown at the right. Consider the horizontal planes represented in the drawing. One plane just touches the sphere at the North Pole, and one just touches the sphere at the South Pole. A plane, which just touches, or intersects the sphere at one point is said to be tangent to the sphere at that point. At each point of a sphere there will be a plane tangent to the sphere at that point.







Suppose we lower the horizontal plane as shown in the second drawing at the right. Will the plane intersect the sphere in just one point? The intersection set of a sphere and a plane as shown at the right will be a circle. What happens to the length of the circle as we move the plane lower? The circles will increase in size until we reach the "middle" of the sphere. We call this circle at the middle the *equator*. From that position, we move the plane lower, the circles decrease in length until the intersection set again consists of one point at the South Pole. Thus, a plane which intersects a sphere may have an intersection set with the sphere which is a circle or consists of only one point. Of course, it is also possible that the plane and sphere do not intersect at all. Then the intersection set is the empty set.

Suppose we consider the set of circles on the surface of the earth which are intersections of the earth with planes parallel to the equator. In this set, the equator has two properties which none of the other circles of the set have. First, its length is greater than the lengths of the other circles. Second, its plane passes through the center of the sphere. We call the equator a *great circle* and the other circles *small circles*.

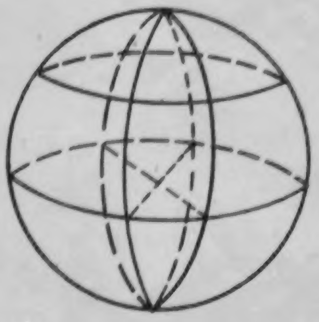
Of course, any plane through the center of the sphere cuts the sphere in a circle of the same maximum size. Any one of these circles is a great circle.

Definition. *A great circle on a sphere is any intersection of the sphere with a plane through the center of the sphere*.

Definition. All circles on a sphere which are not great circles are called small circles.

All great circles on a sphere have the same length since their radii are equal to the radius of the sphere. The length of every great circle on a sphere is greater than the length of any small circle on that sphere.

Again, think of the earth as a sphere. We can imagine many great circles of this sphere. A particular set of great circles of the earth is the set consisting of those great circles which pass through the North Pole and the South Pole. On such a great circle, consider the half-circle which runs from one pole to the other. Such a half-circle with the poles as end-points is called a meridian. We sometimes use the term "semi-circle" in talking about half of a circle. Thus, all meridians are semi-circles.



The small circles whose planes are parallel to the plane of the equator are called parallels of latitude. Each parallel of latitude has its center on the axis of the earth and its plane perpendicular to the axis of the earth.

You know that when you face north, east will be to your right, west to your left and south at your back. On the surface of the earth, east and west from the point where you stand will be along the parallel of latitude through this point. When we want to emphasize that a direction is exactly east, we sometimes say "due east."

Parallel circles of latitude and meridian semi-circles will be discussed more carefully later. At that time we shall discuss how points on the surface of the earth can be located by means of these great and small circles.

You already know that, in a plane, the shortest distance between two points is along a straight line. On a sphere this is not true, although it may appear to be true when you think of two points rather close together on the earth. Can a plane, flying from New York to San Francisco travel along a path which is a straight line? Of course not, it must follow the curvature of the earth. On a sphere, it turns out to be true that the shortest distance between any two points is a path along a great circle that passes through the two points. (You may have heard of "the great circle route" for airplanes and ships.) The proof of this important fact is much too difficult to be given here. However, by using a string stretched around a globe you may test this statement.

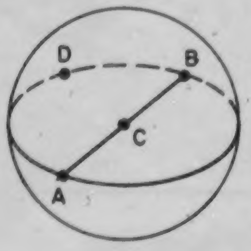
Exercises 12-2

Do your best to answer each question in Problems 1 to 6. When asked to explain your answer give what reasons you can, but do not feel that you have to prove that your answers are correct. You should be prepared, however, to supply reasons for your answers in most cases. The purpose of these questions is to help you start thinking about some of the properties of spheres. You will find it extremely helpful to make drawings on a large ball or some other spherical object. Such drawings will help you "see" the things we talk about.

1. (a) Is there an antipode of any given point on a sphere?

   (b) Is there more than one antipode of any given point on a sphere?

2. In the drawing at the right, C is the center of the sphere. A and B are antipodal points. The great circle passing through A and B is shown as a curve. D is a point on this great circle passing through A and B. The part of the great circle not seen from the front is shown as dotted lines.



(a) Measuring along a straight line through the interior of the sphere which distance is greater, AB or AC?

(b) Measuring along the surface of the sphere, is it farther from A to D or from A to B?

(c) Is there any point on this great circle farther from A than B, measuring either on a line through the interior or along the great circle on the sphere? Explain.

(d) Do all great circles containing A pass through point B? Explain.

(e) Do any other great circles passing through A also pass through point D? Explain.

3. (a) How many great circles pass through a given point, such as the North Pole, of a sphere?

(b) How many small circles pass through a given point of a sphere?

(c) Can a small circle pass through a pair of antipodal points on a sphere? Explain.

4. (a) On a sphere, does every small circle intersect every other small circle? Explain.

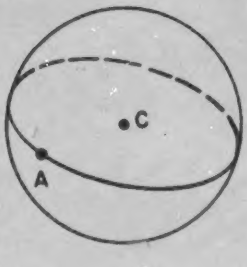
(b) On a sphere, does every great circle intersect every other great circle? Explain.

5. (a) In how many points does each meridian cut the equator? Explain.

(b) In how many points does each meridian cut each parallel of latitude?

(c) Does a parallel of latitude intersect any other parallel of latitude? Explain.

6. In the drawing shown at the right, C represents the center of a sphere with point A on the surface of the sphere.



   (a) Assume that D is a point between A and C on $\overline{AC}$. Where is D located with respect to the sphere?

   (b) E is a point on $\overleftrightarrow{AC}$ but is not on $\overline{AC}$. Must E be inside the sphere?

   (c) B is a point on $\overleftrightarrow{AC}$ and is on the sphere. What is the relation between points A and B? Explain.

   (d) Assume a point E is the antipode of point A. Will E be on $\overleftrightarrow{AC}$? Explain.

   (e) If $\overline{AB}$ is a diameter of the sphere, what name can we give to $\overline{AC}$ or $\overline{BC}$?

7. In the chapter on Circles in Volume I, the interior of a circle of radius r and center C was defined as the set of all points at a distance less than r from the center C. (This set includes the center C itself.)

   (a) Using the above definition as an example, define the interior of a sphere.

   (b) Similarly, define the exterior of a sphere.

   (c) Similarly, define a sphere.

12-3. Properties of Great Circles

Can you remember when you learned that the earth is spherical? Some young children are amazed when they learn that the earth is spherical. This is not strange, however, for we know that at the time Columbus was living many intelligent adults believed the earth was generally flat, much like a plane. Suppose this were true. How then could a person go "around the world"?

We use the term "navigate" to describe the process of directing our movements on earth. Deciding what course, or direction, a ship or airplane is to follow is the responsibility of the navigator. To avoid unnecessary expense, the navigator must try to direct the movement of the ship, or airplane, along the most direct route between stopping places.

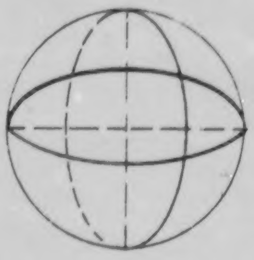
Suppose you want to take a short trip. By a short trip we mean keeping within the boundaries of one of the smaller states, such as Connecticut. For such a trip you can use a road map as a guide. Most road maps are flat, like planes. Let us assume that the actual surface of the country is flat and that the roads joining cities are all straight segments. Or, we can assume you will use a small plane and can travel along straight line segments. For such a trip it would be easy to decide on the shortest route, and it would be easy to decide which direction to follow.

However, if you are planning a trip from San Francisco to London or from New York to Buenos Aires, a road map is of little help. For planning such a trip, a model of the earth, such as a globe, would usually be more helpful than a map in the form of a plane. You will sometimes find that things are not what they seem to be.

To understand travel on the globe better, let us review some fundamental properties of spheres. In Exercises 12-2 two fundamental ideas about spheres were introduced. These ideas deal with great circles. The first idea may be stated as a property.

Property 1. Every pair of distinct great circles intersect in two antipodal points.

This property is easily proved as follows:



(1) Every great circle of a sphere lies on a plane through the center of the sphere.

(2) All planes containing a great circle have the

center in common, and thus any two such planes must intersect.

(3) The intersection set of any two intersecting planes is a line.

(4) This line intersects the sphere in two antipodal points since the line passes through the center.

(5) Thus, these antipodal points are the points of intersection of the two great circles on the two planes.

We will use this property in discussing distances between points on a sphere.

In the previous section we stated that the shortest distance on the surface of a sphere between any two points on the sphere is measured along the path of a great circle. In the study of geometry in high school mathematics this statement is proved. We will not do so now but we shall use the result as a fact.

Assume you are to find the shortest route between two points on the globe. Suppose you are to travel from the North Pole to the South Pole. Is there one shortest route? No, for you can easily find as many "shortest" routes as you please. Each meridian is a possible route. If we think of the earth as a sphere, the meridians are congruent. Thus, it does not matter which meridian is selected as your route. For any two antipodes, there are any number of paths one can take, namely any of the great circle paths determined by the two antipodal points.

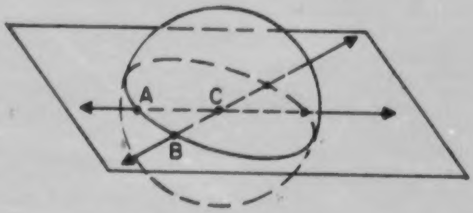
But, what if the two points are not antipodes? How many possible paths along a great circle route are there? We can show that there are only two possible great circle paths between two such points. Moreover, both these paths lie on the same great circle. This is the next important property.

Property 2. Through any two points of a sphere, which are not antipodes, there is exactly one great circle.

We can prove this property as follows:

(1) On a sphere consider any two points, A and B, which are not antipodes.

(2) Since A and B are not antipodes, a line through A and the center of the sphere, C cannot pass through B. (Similarly, $\overrightarrow{BC}$ does not contain A.)

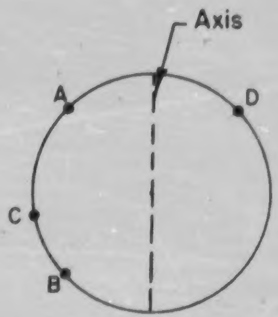


(3) Through the three points A, B, and C there can pass exactly one plane, because these three points are not contained in one line.

(4) The one plane containing A, B, and C, contains only one great circle with center at C.

(5) This great circle is thus the only great circle passing through A and B.

This property tells us that if two points on the sphere are not antipodes, there is exactly one shortest route between these two points. Of course, there are two directions one can travel along a great circle containing A and B. In the drawing at the right we see that one route, ADB, would pass through D, the other, ACB, through C. Since A and B are not antipodes, one route must be shorter than the other. We naturally choose the one which is shorter. Can you pick the shortest route in the drawing?



From the point of view of shortest distance, the great circles on a sphere behave like straight lines on a plane. We have shown also that through any two points there is just one great circle unless the points are antipodal. But great circles on a sphere do not behave like straight lines in all respects for any two great circles intersect in two points. There are no parallel great circles on a sphere.

Exercises 12-3

Use a globe and length of string and a ruler to answer Problems 1 to 3.

1. Locate Nome, Alaska and Stockholm, Sweden on the globe.

   (a) Place one endpoint of the string on the location of Nome. Place the string on a northern path, passing through the North Pole. Continue until you reach Stockholm. Carefully mark on the string a point which falls on the location of Stockholm. What is the distance on the globe in inches from Nome to Stockholm as represented by the segment marked on the string?

   (b) Using a string and ruler, what is the distance from Nome to Stockholm along a route directly east from Nome?

   (c) From your results above, what is the shorter distance between the two points represented by Nome and Stockholm: a path following a great circle; or a path following the small circle which is the parallel of latitude?

2. (a) What is the distance from Nome to Rome along a great circle route which passes through a point near the North Pole?

   (b) What is the distance from Nome to Rome along a southeasterly course passing through the southern tip of Hudson Bay, and through a point on the border between Spain and France?

   (c) How do your results in (a) and (b) compare?

3. A merchant living in Singapore, Malaya, plans to take a non-stop flight to Quito, Ecuador. What is the best route between these two points?

4. Explain why going due north would be the shortest although not necessarily the safest or best, route in traveling to a point on the earth directly north of your starting point.

5. (a) Explain why going due east is not always the most efficient way of getting to a point directly to the east.

   (b) When is a route due east or west always the most efficient?

*6. Given three points on a sphere. Can a circle on the sphere (small or great) be drawn through all three points?

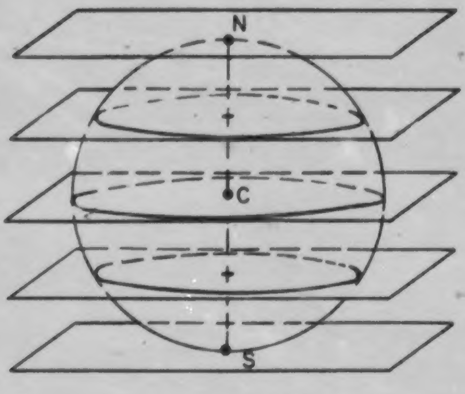
7. BRAINBUSTER: A hunter set out walking due south from his camp. He walked for about two hours without seeing any game. Then he walked 12 miles due east. At this point he saw a bear which he shot. To return to camp he traveled directly north. What color was the bear? (Note: This problem does have an answer.)

12-4. Locating Points on the Surface of the Earth

You already know how to locate points in a city by the streets and street numbers. Suppose, however, that you are planning to visit a friend who has just moved to a farm. You need some instructions to find the location of your friend's new home. You might be told: "Start from where you are now. From this point, go east 2 miles, then go north for 1 mile." These instructions give you three things: a starting (or reference) point, the directions you must follow from the starting point, the distances you must travel in those direction.

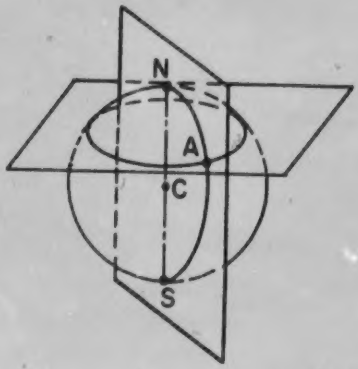
Imagine that you are a pilot of a transoceanic airplane, or the captain of an ocean ship. It is your responsibility to help navigate, or direct the course, of the ship or plane. The course will follow various headings, or "paths." Before any directions for the headings have any meaning, you must know where you are. That is, you must know your location on the earth. By keeping track of the location of the plane or ship on maps, it is possible to determine the correct headings. Your first job is to find your location. How can this be done?

To locate points on the earth's surface, we think of the earth as a sphere and define two sets of curves on the sphere. One set of curves consists of the circles called parallels which we mentioned in Section 2. There we had a set of parallel planes slicing the earth in horizontal sections as shown at the right. The top plane is tangent to the North Pole, and the bottom plane is tangent to the South Pole. The intersection of each of the remaining planes and the earth is a circle. The circles determined by these planes are all small circles except for the equator, which is a great circle. All such circles are called parallels of latitude. They are called "parallels" because they are determined by planes parallel to the plane passing through the equator.



The second set of curves consists of the meridians, which also have been described earlier in this chapter. Remember that meridians are halves of great circles which have the poles as endpoints. Thus, each great circle through the poles consists of two meridians. Each meridian has as its diameter the axis of the earth.

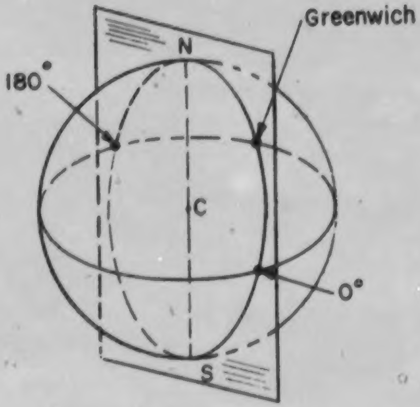
Let A be some point on the sphere as shown below. There is exactly one plane through A, perpendicular to the axis of the earth. This plane contains the parallel of latitude through A. There is exactly one meridian through A because the point A and the North Pole (or South Pole) determine one great circle. Since that great circle passes through the poles, the arc of the great circle containing A is a meridian. Thus, through each point of a sphere, except the poles, there is exactly one parallel of latitude and one meridian.



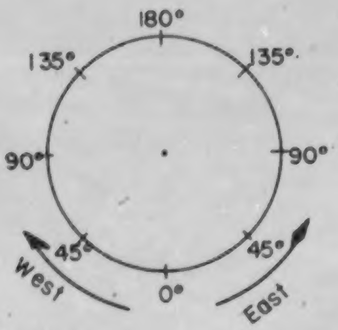
It remains only to find numerical names for the meridians and parallels of latitude. How can we do this? We select one of the meridians on the earth as a reference line. We label this the *zero meridian*. The other meridians are east or west of the zero meridian, just as streets in a town are east or west of the street used as a reference.

Actually, the zero meridian for the earth has been designated. It is the meridian which passes through a certain location in Greenwich (pronounced Gren - ich), England. Greenwich is near London. We sometimes refer to this meridian as the *Greenwich meridian*, even though the meridian itself passes through one particular point of the town. The meridian at Greenwich is sometimes called the prime meridian. (This has nothing to do with a prime number.)

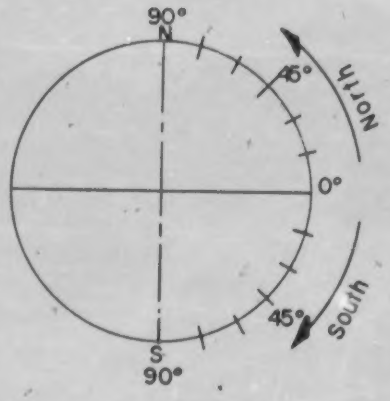
The intersection of the Greenwich meridian and the equator is marked 0°. From this point, we follow the equator east, or west, until we reach the meridian which passes through the antipode of the Greenwich point that lies on the great circle through Greenwich and the North Pole. This meridian intersects the equator at a point which is half way around the equator from the point labeled 0°. This point is labeled 180°. We can think of a plane intersecting the earth in this great circle. The plane separates the earth into two hemispheres, or half-spheres, as shown in the drawing at the right. The hemisphere on the left as you look at the drawing, is named the western hemisphere. The hemisphere on the right is the eastern hemisphere.



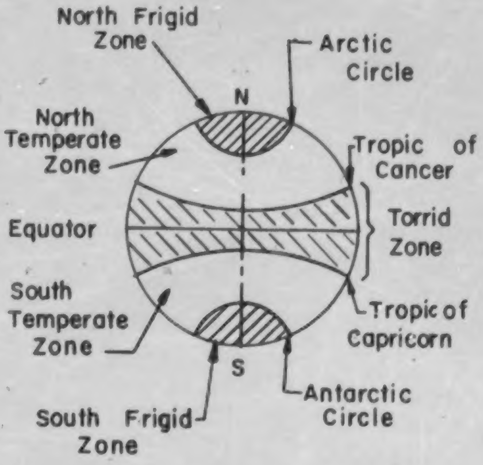
The great circle, which we call the equator, is divided into 360 equal parts as shown at the right, as seen by an astronaut above the North Pole. The numbers between 0 and 180 are assigned to the points on the half equator to the left of 0°. The same is done for the points on the other half of the equator. Each of these points names the meridian passing through that point. Any point on earth may be located by the meridian passing through the point. For example, Los Angeles is approximately on the meridian 120° west of the Greenwich meridian. Tokyo is approximately on the meridian 140° east of Greenwich. We say the longitude of Los Angeles is about 120°W. (west). The longitude of Tokyo is about 140°E.



The parallels of latitude are located in the following way. The equator is designated the zero parallel. All points above the equator are in the northern hemisphere, points below in the southern hemisphere. We choose any meridian, for instance that meridian through Greenwich. The part of the meridian from the intersection with the equator to the North Pole is divided into 90 equal parts, assuming that the earth is a sphere. The whole numbers between 0 and 90 are assigned to these points. Each point determines a parallel of latitude. A similar pattern is followed for points on the meridian south of the equator. For any point on earth, we may locate the parallel of latitude containing the point. For example, New Orleans is approximately on the parallel 30° north of the equator. Wellington, New Zealand, is approximately on the parallel 40° south of the equator. We say that New Orleans has a *latitude* of about 30°N. (north). The latitude of Wellington is approximately 40°S.


90°
N
45°
North
0°
45°
South
S
90°

Some of the parallels are given special names. The *Arctic* and *Antarctic Circles* are the parallels located about $23\frac{1}{2}$ degrees from the North and South Poles. The *Tropic of Cancer* is about $23\frac{1}{2}$ degrees north of the equator, and the *Tropic of Capricorn* is about $23\frac{1}{2}$ degrees south of the equator. Portions of spheres between two parallels of latitude are sometimes called *zones*. Some of these zones are also given special names as shown in the drawing at the right. Can you locate the zone in which you live? Can you name the hemisphere in which you live? Could there be two different correct answers?


North Frigid Zone
Arctic Circle
N
North Temperate Zone
Tropic of Cancer
Equator
Torrid Zone
South Temperate Zone
Tropic of Capricorn
S
Antarctic Circle
South Frigid Zone

To locate a point on earth, we name the meridian and the parallel of latitude passing through the point. Thus we name the longitude and the latitude of a point. For example: 90°W., 30°N. locates a point in the city of New Orleans. We say that New Orleans is located approximately at this point on earth. Durban, South Africa, is located at approximately 30°E., 30°S. Note that the longitude is always listed first. Do the longitude and latitude of a point help you locate the hemisphere in which the point is located? Notice that latitude and longitude give a coordinate system on the sphere much as the X-axis and Y-axis give a coordinate system in the plane.

Exercises 12-4

1. Using a globe, find the approximate location of each of the following cities. Indicate the location by listing the longitude first, followed by the latitude. Be sure to include the letters E or W and N or S in your answers.

   (a) New York City
   (b) Chicago
   (c) San Francisco
   (d) London
   (e) Paris
   (f) Moscow
   (g) Rio de Janeiro
   (h) Melbourne, Australia

2. Greenwich, England, is located approximately on the parallel of latitude labeled 52°N. Without getting further information, write the location of Greenwich.

3. Chisimaio, Somalia, in eastern Africa, is located on the equator (or very near the equator). It is about 42 degrees east of Greenwich. Without using a reference, write the location of Chisimaio.

4. Find answers to the following, using such reference materials as encyclopedias, maps, and social science and history books:

   (a) What is the parallel separating North from South Korea?

(b) Find some states, parts of whose boundaries are along parallels of latitude.

(c) What parallel of latitude was involved in the dispute between the United States and Great Britain about their boundary in the Northwest?

(d) What parallel of latitude was connected with the Missouri Compromise?

(e) What parallel of latitude is associated with the Mason and Dixon line?

(f) Can you guess the reason for the name of the country Ecuador in South America?

5. Using a map or a globe, find cities located approximately at the following longitudes and latitudes:

   (a) 58°W.,35°S. (b) 175°E.,41°S.

6. Using a map or a globe, find cities located approximately at the following longitudes and latitudes:

   (a) 122°E.,35°N. (b) 5°W.,41°N.

7. (a) Compare the location of the city in your answer for 5(a) with the location of the city in your answer for 6(a).

   (b) Similarly, compare the locations of the cities determined by 5(b) and 6(b).

   (c) What kind of points do these locations suggest?

8. (a) Determine the location of your home town.

   (b) Determine the antipodal point of this location.

   (c) If you could travel from a spot in your home town through the center of the earth, would you come out in China?

9. What point in the United States is closest to Moscow, Russia?

10. Using a map or other reference, find the answers to the following questions:

    (a) Is Reno, Nevada, east of Los Angeles? That is, does the meridian through Reno lie east of the meridian through Los Angeles?

    (b) About where would the meridian through Miami, Florida, strike South America?

    (c) Which of the following cities is closest to being on the same parallel of latitude with New York City: San Francisco; Portland, Oregon; or Seattle, Washington?

    (d) Which of the following cities is closest to being on the same parallel of latitude with New York City; London; Madrid; or Casablanca, Morocco?

*11. Are there two different points on the earth which have the same latitude and longitude? If so, where are they, and if not, explain why there are none.

*12. Are there any points on the earth that have more than one location in terms of latitude and longitude? Explain why, or why not.

*13. Determine a way of finding the location of an antipodal point, say of 90° W., 45° N., without using a globe or map. Then find the antipodal points of each of the following:

    (a) 80° W., 25° S.

    (b) 100° E., 65° N.

    (c) 180° W., 52° S.

*14. Find the reasons for the location of the Arctic Circle and the Tropic of Cancer.

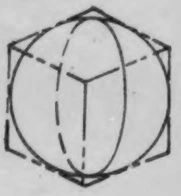
*15. Where and what is the International Date Line?

*16. Southeast of Australia, there is located a group of islands called the Antipodes Islands. They received this name because they are antipodal to Greenwich. Without using a reference, write the location of the Antipodes. When it is midnight in Greenwich, what time of day is it at the Antipodes? When it is the middle of summer in Greenwich, what season is it in the Antipodes? Does this mean that when it is June 21st in Greenwich it is December 21st in the Antipodes?

12-5. Volume and Surface Area of a Spherical Solid

In previous chapters a cube has been discussed in detail. Remember that it is a surface. The volume of a cube is the volume of the rectangular solid whose surface is a cube. If the measure of an edge of the cube is e then the measure, V_c, of the volume is e^3. The volume of the cube is e^3 cubic units. The subscript, c, merely means that this refers to the cube.

A sphere is a surface. By the volume of a sphere we will mean the volume of the solid whose surface is a sphere. In this section we give a formula for the measure, V_s, of the volume. Let r stand for the radius (that is, the number of units in the length of the radius) of the sphere and see first how we can get an approximate idea of its volume. Build around this sphere the smallest possible cube. This cube should touch the sphere as indicated in the diagram. It looks like a baseball in a tight-fitting box.



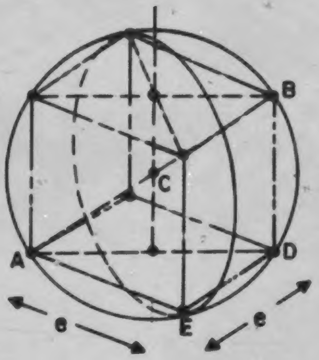
The edge of the cube has measure $2r$. Hence

$$V_c = (2r)^3 = 8r^3.$$

Since the sphere lies entirely within the cube, the volume of the sphere is less than the volume of the cube. Hence

$$V_s < 8r^3.$$

Now think of constructing a cube entirely within the sphere so that all vertices of the cube lie on the sphere. The points A and B are shown as opposite vertices of the cube, and C is the center of the sphere and of the cube. You see that C lies on the segment $\overline{AB}$. Hence A and B are antipodes. ADE is a right triangle, so



$$(AD)^2 = e^2 + e^2 = 2e^2$$

where e is the number of units in AE. Now ADB is a right triangle and hence

$$(AD)^2 + (BD)^2 = (AB)^2.$$

But BD = e and AB = 2r. Hence

$$2e^2 + e^2 = (2r)^2,$$

$$3e^2 = 4r^2,$$

$$e^2 = \frac{4}{3} r^2 = \frac{4 \cdot 3}{3 \cdot 3} r^2$$

$$e \cdot e = \left(\frac{2\sqrt{3}\ r}{3}\right) \left(\frac{2\sqrt{3}\ r}{3}\right)$$

$$e = \frac{2\sqrt{3}\ r}{3}$$

Hence, $V_c = e^3 = e^2 \cdot e = \frac{4}{3} r^2 \cdot \frac{2\sqrt{3}}{3} r = \frac{8\sqrt{3}}{9} r^3 \approx 1.53r^3.$

Since the volume of the sphere is larger than the volume of this cube,

$$V_s > 1.53r^3.$$

Therefore,

$$1.53r^3 < V_s < 8r^3.$$

We have obtained two numbers between which the measure of the volume of the sphere lies. Actually these numbers are not very close to each other as you can see by letting $r = 2$ and computing the volumes of the two cubes. To find an accurate formula for the volume of a sphere is a task far too difficult at this time. But, it can be proved that the volume of a spherical solid is $\frac{4}{3}\pi r^3$ units, where π is the same number we met when working with circles, and r is the measure of the radius of the sphere. Notice that $\frac{4}{3}\pi \approx 4$, so that $\frac{4}{3}\pi r^3$ certainly lies between $1.53r^3$ and $8r^3$. The number π is a real number though not a rational number; it is an irrational number. Its decimal form has been computed to many places. To five decimal places it is

$$\pi = 3.14159 \cdots$$

Although we do not prove it in this book, we shall use the fact that the volume, V_s, of a sphere with radius of measure r is given by the formula

$$V_s = \frac{4}{3}\pi r^3 \text{ cubic units.}$$

If the radius of the sphere is measured in inches, then the volume is measured in cubic inches. Other units are handled in similar fashion.

Surface Area of a Sphere

We were able to approximate the volume of a sphere by comparing it to the volume of a smaller cube inside the sphere and to a larger cube containing the sphere. We now ask how to get some rough estimate of the surface area of the sphere (the area of the skin of the orange, so to speak).

If we look back at the cube which we constructed to enclose the sphere, we see that the edge of the cube has measure $2r$. Hence, each face of the cube has area $(2r) \times (2r) = 4r^2$. There are 6 faces on the cube. Therefore, the total surface area of the cube has measure $6 \times 4r^2 = 24r^2$. It seems clear that the area of the sphere is less than the surface area of this cube which encloses it.

Thus,

$$A_s < 24r^2$$

where A_s represents the area of the sphere of radius r.

Look now at the surface of the enclosed cube. We showed earlier that the square of its edge e was

$$e^2 = \frac{4}{3} r^2.$$

But e^2 is just the measure of the area of each face of the inner cube. Then the surface area of this enclosed cube is

$$6 \times \frac{4}{3} r^2 = 8 r^2.$$

Since the surface area of the sphere is greater than the surface area of this cube which lies entirely inside the sphere, we have

$$8r^2 < A_s.$$

From these two inequalities we can write

$$8r^2 < A_s < 24r^2.$$

Here, as in the case of the volume of the sphere, we do not have limits which are very close together, but they do give a very rough idea of the area and the method suggests how to go about getting a better approximation to the surface area. You will prove in one of your later courses that:

> The area of the surface of a sphere is $4\pi r^2$ square units of area, where r is the measure of the radius of the sphere. If A_s represents the surface area, then
>
> $$A_s = 4\pi r^2.$$

Note that $A_s = 4\pi r^2$ is approximately $A_s = 12.6\, r^2$ which is within the range $8r^2$ to $24r^2$ we found earlier for A_s.

We may think of this surface area in comparison to the area enclosed by a great circle of the sphere. A great circle of the sphere will have radius r and hence an area of πr^2 square units. Thus,

$$A_s = 4 \times \pi r^2 \quad \text{or,}$$

$$\begin{Bmatrix} \text{surface area} \\ \text{of a sphere} \end{Bmatrix} = 4 \times \begin{Bmatrix} \text{area of one of its} \\ \text{great circles} \end{Bmatrix} .$$

If the radius of a sphere is measured in inches, its surface area is measured in square inches.

Spherical Soap Bubbles

Let us now develop ideas which will assist us in discussing the soap bubble. Suppose, first, we consider a sphere of radius 2 units. Then our formulas give us

$$A_s = 4\pi \cdot 2^2 = 16\pi \quad \text{and} \quad V_s = \frac{4}{3}\pi \cdot 2^3 = \frac{32}{3}\pi$$

for the area and volume of the sphere respectively. Suppose we have a cube with the same volume as this sphere. How would the area of the cube compare with that of the sphere? The volume of the sphere is a little more than 32, since π is a little more than 3. Hence, if e is the number of units in the edge of the cube, then e^3 must be a little greater than 32, since e^3 is the volume of the cube. As a result, e is a little larger than 3. In other words, the length of the edge of the cube having the same volume as the sphere will be a little more than 3 units. Since the cube has six faces, $A_c = 6e^2$. Hence its area will be greater than $6 \cdot 3^2 = 54$. This number is certainly greater than 16π, the surface area of the sphere. If we had done this work more accurately, the areas would be found to be:

$$A_s \approx 50.26, \quad A_c \approx 62.48,$$

which reveals a somewhat greater discrepancy.

Suppose we try another example and consider the sphere of radius 5 units. Then our formulas give us

$$A_s = 4\pi \cdot 5^2 = 100\pi \quad \text{and} \quad V_s = \frac{4}{3}\pi \cdot 5^3 = \frac{500}{3}\pi \ .$$

Here V_s is about 527 and so, for a cube of the same volume, $V_c \approx 527$. Hence $e^3 \approx 527$, so that e is a little more than 8 since $8^3 = 512$. Accordingly, A_c is a little more than $6 \cdot 8^2 = 384$. In this example, $A_s \approx 314.16$ and $A_c \approx 384$. Again the area of the sphere is decidedly less than the area of the cube of equal volume. These are two examples of the following fact:

> If a sphere and a cube have equal volumes, the surface area of the sphere is the smaller.

In fact, a stronger statement is true:

> If any solid has the same volume as a sphere, the surface area of the sphere is less than or equal to the surface area of the solid.

Now to our soap bubble! The soap film has a certain elasticity to it. This elasticity "pulls in" the film as much as possible. A given volume of air is trapped inside the bubble. Hence the physical property requires the surface to have as small an area as possible for the given volume. As stated in the preceding paragraph this area will be least when the surface is a sphere. This is why soap bubbles are spherical in shape.

Exercises 12-5

1. For each sphere whose radius is given below, find the volume of the corresponding spherical solid. Use $\frac{22}{7}$ as the approximation for π.

   (a) $r = 3$ inches
   (b) $r = 10$ feet
   (c) $r = 4$ yards
   (d) $r = 6$ cm.
   (e) $r = 5.6$ inches
   (f) $r = 6.6$ inches
   (g) $r = 8.4$ mm.
   (h) $r = 4.2$ feet

In Problems 2, 3, 5, and 6 use 3.14 as the approximation for π.

2. For the various spheres of Problem 1 find the surface area.

3. An oil tank is in the shape of a sphere whose diameter is 50 feet. The tank rests on a concrete slab.

   (a) If paint costs $6 per gallon and a gallon covers 400 square feet, find the cost of the paint for the surface.

   (b) If oil costs 13 cents per gallon find the value of the oil in the tank. Assume that the tank is full. (1 cu. ft. $\approx 7\frac{1}{2}$ gallons)

4. Suppose your mother has a bowl in the shape of a hemisphere of radius 8 inches. She borrows a bowlful of sugar and wishes to pay for the sugar rather than return it. Assume that sugar costs 10 cents per pound and that 1 pound occupies 32 cubic inches. Use $\pi \approx 3$. How much does your mother pay?

5. A mapmaker wishes to produce 100 globes made of plastic sheeting. The diameter of each globe is to be 18 inches. Find the cost of the globes if the plastic costs 50 cents per square foot.

6. A spherical balloon has a diameter of 40 feet. How much gas will it hold when all the air has been pushed out?

7. (a) If the radius of a sphere is doubled what effect does this have on the volume? On the surface area?

   (b) If the radius of a sphere is tripled what effect does this have on the volume? On the surface area?

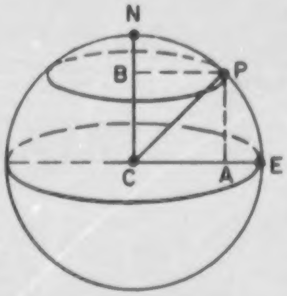
*8. Two spheres have radii in the ratio $\frac{3}{2}$.

   (a) Find the ratio of their volumes.

   (b) Find the ratio of their surface areas.

12-6. Finding Lengths of Small Circles

How would we find the length of a circle of latitude? In this section we will show how this may be done using values of cosines of angles. First draw a picture of the earth. Call N the North Pole, C the center of the earth, P some point on the surface of the earth and E the point directly south of P on the equator. The figure shows the great circle through P and E. Then the measure in degrees of the angle PCE is the latitude of point P. Let A be chosen so that PAC is a right angle and choose B on the axis of the earth so that PBC is a right angle. Then PBCA is a rectangle and hence $\overline{PB}$ is congruent to $\overline{CA}$, that is, the lengths PB and CA are equal. But $\frac{CA}{CP}$ is equal to the cosine of angle PCA. Thus if L° is the latitude of P, we have



$$\frac{CA}{CP} = \cos L^\circ.$$

The equator is the great circle shown in the diagram as being in a plane perpendicular to the paper and passing through E. The radius of the circle is CE. Call e the length of this great circle. Hence, the length of the equator is $2\pi(CE) = 2\pi(CP)$, or

$$e = 2\pi(CP).$$

Why is $CP = CE$? The circle of latitude at P has center at B, has radius BP and has its plane perpendicular to $\overline{NC}$. Denote the measure of the length of the circle of latitude by p. Hence

$$p = 2\pi(BP) = 2\pi(CA).$$

Accordingly,

$$\cos L^\circ = \frac{CA}{CP} = \frac{2\pi(CA)}{2\pi(CP)} = \frac{p}{e}.$$

What property justifies the second equality in the last line?

Since $\cos L^\circ = \frac{p}{e}$, it follows that $p = e \cos L^\circ$.

The length of the equator is about 25,000 miles. This gives

$$p \approx 25{,}000 \cos L^\circ,$$

where L° is the latitude of the point P. If the latitude of a point is known, the length of the small circle through the point can be found by using values of the cosines of various angles given in the table of Chapter 9.

Example. The latitude of a certain city is 35°. Find the approximate length of the small circle through the city. The length of the small circle is about $25{,}000(\cos 35^\circ)$ miles, that is $25{,}000(0.8192)$ miles, which is 20,480 miles.

Exercises 12-6

1. Find to the nearest ten miles the length of the circle of latitude which passes through the point with latitude given below.

   (a) 15° (b) 75° (c) 45°

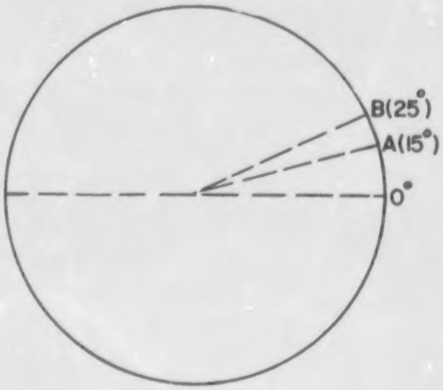
2. City A has longitude 15°E. and city B has longitude 25°E. The cities are on the same parallel of latitude, 30°N.

   (a) Find the length of the circle of latitude on which the cities lie.

   (b) The diagram represents the circle of latitude. Find the length of the semicircle OBC.

   (c) Find the difference between the longitudes of A and B.

   (d) Length of arc AB is __?__ (fractional part) of the length of the semicircle.



(e) Find the length of AB.

(f) An airplane travels close to the surface of the earth and follows the parallel of latitude from A to B. If it travels 400 m.p.h., how long does it take for the journey?

3. How far is it between meridian $10^{\circ}W.$ and $70^{\circ}W.$ at latitude $40^{\circ}N.$ along the parallel of latitude?

*4. By sun-time is meant the time as determined by the position of the sun. Standard time zones should not enter into this problem.

(a) If sun-time is 7:00 a.m. at meridian $10^{\circ}W.$, find the sun-time at $70^{\circ}W.$

(b) If sun-time is 7:00 a.m. at meridian $70^{\circ}W.$, find the sun-time at $10^{\circ}E.$

*5. Cities A and B both have the latitude $40^{\circ}N.$ and are in the same time zone, that is, a person does not change his watch in going from one to the other. The sun rises exactly one hour later at A than at B. How far apart are the cities and which direction is A from B?

6. OPTIONAL. In section 12-5 we performed some computations to help us determine the volume of a sphere. These computations gave us only approximate results. It was stated that the correct formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$. It was also stated that we would not attempt to prove this formula at this time. Later, in more advanced courses in mathematics, you will learn to prove that the formula given is correct.

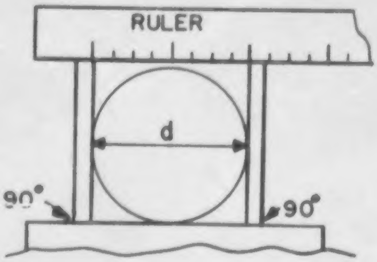
Try the following experiment to verify that the formula for the volume of a sphere is correct. All measuring for this experiment must be carefully done.

OBJECTIVE: To verify that the formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$.

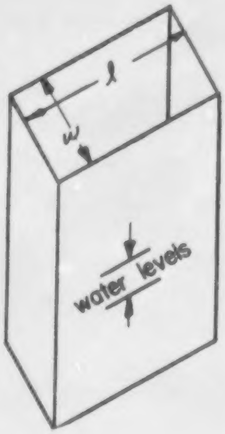
MATERIALS:
(a) A sphere (a softball will do nicely)
(b) A container (one having an interior shaped in the form of a rectangular solid, such as a half-gallon, paper milk carton with top cut off)
(c) A ruler (graduated in 16ths of an inch or in tenths of a centimeter)
(d) A container with about one quart of water.

DIRECTIONS:

(1) Measure the diameter as shown at the right. Compute the radius of the sphere.



(2) Measure the length and width of the interior of the container. Partially fill the container with water. Mark the water level carefully. Immerse the sphere in the water so that it is completely covered with water. Carefully mark the new water level. Find the measure of the distance the water level was raised.



(3) Find the product of the measures of the following: the length of the interior of the container; the width of the interior of the container; the distance the water level was raised.

(4) Is the answer to (3) the volume of the sphere? Explain why, or why not.

(5) Compute the value of r^3, using the radius of the sphere. Is r^3 less than the volume of the sphere?

(6) Find the product of π and r^3. Is this answer less than the volume of the sphere?

(7) Divide the answer for (3) by the answer for (6). If you did all of your work carefully, you should obtain an answer approximately equal to $1\frac{1}{3}$ or 1.3. Did you? How does $1\frac{1}{3}$ or 1.3 compare with $\frac{4}{3}$?

Chapter 13

WHAT NOBODY KNOWS ABOUT MATHEMATICS

13-1. Introduction

With daily articles and newspaper reports about rockets, weather satellites, and radio telescopes, to say nothing of new antibiotics and space medicine, everyone is keenly aware of the active work and the many unsolved problems in different branches of science. Many people, however, have the curious idea that mathematics is a dead and completed subject that was embalmed between the covers of a textbook sometime after Sir Isaac Newton. Actually mathematics is as active as any of the fields just mentioned and has a wide variety of challenging problems not yet solved.

Not only the amount of new mathematics but the number of kinds of mathematics is increasing at breathtaking speed. At the close of this chapter is reproduced the subject classification for the Mathematical Reviews, a publication which contains brief summaries of mathematical research and runs to over 1000 double-column pages a year. There are 436 topics in the list, which is more than four times the number of topics in the previous classification in 1958. Most of these names are unfamiliar to you. You may associate new mathematics with computing, an important new branch of the subject; but you will notice that "Computing Machines" under "Numerical Analysis" occupies only a small portion of the list.

Some of the topics have familiar names, like geometry and algebra, but you will find few familiar names under these headings. A person who graduates from college or even who has a Ph.D. degree in mathematics cannot hope to have any very deep knowledge of more than a few of the topics there listed. Most of the applications of mathematics are not even considered in this list but appear in other journals.

In this chapter you will become acquainted with a very few selected unsolved problems in mathematics. These are, of course, not really typical of unsolved problems since we must choose ones which will have some meaning to you and in which we can make a little progress. But even most of the results we state are found by much more advanced methods than you know; how the problems will finally be solved, or whether they will be solved, no one knows.

13-2. A Conjecture About Primes

In this section and the following, some problems having to do with prime numbers are considered. The mathematician is chiefly interested in these for himself, but you may be surprised to know that even here there are applications. Some of the properties of prime numbers are used, for instance, in checking the accuracy of programs set up on computers. Such properties are also used to find sequences of so-called "random digits." Roughly speaking, a sequence of random digits is a sequence of numbers between 0 and 9, inclusive, in which there is no pattern. A repeating decimal, for instance, would have a definite "pattern" but there would seem to be no pattern in the decimal for $\sqrt{2}$.

You have worked with prime numbers before. You remember that a prime number is a counting number greater than 1 which has no factors except itself and 1. Very likely you have used a method called the Sieve of Eratosthenes to find the prime numbers up to one hundred. At the end of this chapter you will find a list of prime numbers up to 1000. Most university libraries contain a list by D.N. Lehmer of primes up to ten million. ("List of Prime Numbers from 1 to 10,006,721" by D.N. Lehmer, Carnegie Institution of Washington, 1914.)

The prime numbers have many interesting properties. One important one which you have used frequently is that every counting number except 1 is either a prime number or can be written as a product of prime factors in only one way except for the order of the factors. For example, $38 = 2 \times 19$ and $75 = 3 \times 5 \times 5$. For instance, no matter how you may write 75 as a product of primes, there will be two 5's and one 3. You use the method of factoring a number into prime factors almost every time you express a fraction in simplest terms. One may ask what numbers we can obtain by adding primes. It is easy to show that all counting numbers above 1 can be obtained by adding primes. To show this, suppose first that n is any even number. Then $n = 2k$, where k is some counting number. (Why?) But then $n = 2 + 2 + \dots + 2$, where 2 occurs k times in the sum and since 2 is a prime this shows every even number is a sum of primes. Suppose now that n is any odd number greater than 1. In this case, what kind of a number is $n - 3$? Is it either 0 or an even counting number? Why? If $n - 3$ is 0 then $n = 3$ which is already a prime, while, if $n - 3$ is an even counting number then $n - 3 = 2k$ and

$$n = 3 + 2 + 2 + \dots + 2,$$

where "2" occurs in the sum k times. Thus every counting number is a sum of primes. In fact, we have shown that we can do this using no primes except 2 and 3.

Suppose now we ask a harder question. What counting numbers can be obtained as a sum of exactly two primes? Since the smallest prime is 2, the smallest number that is a sum of just two primes must be 4, because $4 = 2 + 2$. Can we express any counting number 4 or above as a sum of exactly two primes? Try the numbers from 4 to 20. Did you find any that you could not write as a sum of two primes? How many? Which ones were they? Were they all odd numbers?

Can you think of a reason why it is harder to express an odd number as a sum of two primes than an even number? Perhaps the following observations will help. In the sequence of primes the only even number is 2. Why is this? Now if the sum of two counting numbers is an odd number, what can you say about the two numbers? Is it true that one of them must be even and the other odd? Then if an odd number is to be a sum of two primes, one of the primes is even and thus must be equal to 2. Thus the only possible way of writing 11 would be $2 + 9$. But unfortunately 9 is not a prime, and hence 11 cannot be expressed at all as a sum of two primes.

Since we have seen that we do not get all odd numbers as a sum of two primes, let us concentrate just on the even numbers. Let us ask what even numbers greater than or equal to 4 can be expressed as a sum of two primes. As an experiment try completing the table below for the even numbers from 4 to 26.

$4 = 2 + 2$
$6 = 3 + 3$
$8 = 3 + 5$
$10 =$
$12 =$
$14 =$
$16 =$
$18 =$
$20 =$
$22 =$
$24 =$
$26 =$

Were you able to complete the table and express each of these as a sum of two primes? Incidentally how many of the numbers could be written in more than one way as a sum of two primes? The smallest such number is 10, for $10 = 5 + 5 = 3 + 7$.

Exercises 13-2a

1. Continue the above table of the even numbers up to 100. Were you able to express each of them as a sum of two primes?

2. For the even numbers from 4 to 50 find the number of ways in which each can be written as a sum of two primes. Tabulate the results as follows:

Even Number	Number of Ways	Even Number	Number of Ways
4	1	28	
6	1	30	
8	1	32	
10	2	34	
12	1	36	
14	2	38	
16		40	
18		42	
20		44	
22		46	
24		48	
26		50	

On the basis of your experience here would you hazard a guess as to the Possibility of expressing all even numbers above 4 as a sum of two primes? Specifically what would be your guess about the truth of the following conjecture? (A conjecture is just an educated guess.)

"Every even number greater than or equal to 4 can be written as a sum of two primes."

If you think this is probably true you are in good company. This statement is known as the Goldbach Conjecture. Goldbach suggested it in a letter to the great mathematician Euler, asking whether Euler could prove it. Euler was unable to prove it and, since his time, many mathematicians have worked on this problem. However, up to the present no one knows whether the statement is true or false. It is part of the mathematics that no one knows.

Some progress has been made, however. In 1936, the Russian mathematician, Vinogradov, showed that every odd number beyond a certain number can be expressed as a sum of three primes. Why do we say this is getting close to the Goldbach conjecture? Because if the Goldbach conjecture is true, then Vinogradov's Theorem is an easy consequence. The proof would go like this.

Let us suppose that Goldbach's conjecture is true. Also let n be any odd counting number greater than 5. What can you say about the number n - 3? Is it even or odd? Is it greater than 2? According to Goldbach's conjecture what can you conclude about n - 3? If n - 3 = (sum of two primes), then n = 3 + (sum of two primes). Thus, n is the sum of how many primes? This is Vinogradov's Theorem.

Because this theorem and Goldbach's conjecture are so closely related, many mathematicians thought Vinogradov was on the right track. It is not unlikely that some day he will solve the Goldbach problem.

You may wonder how one might go about trying to prove something like Goldbach's conjecture. It is not enough just to test it in a large number of cases, say with a computer, because there are an infinite number of even numbers to test, and we cannot possibly test all of them. We need a general law applying to all even numbers above 2 showing that they can be written as a sum of two primes. If you find such a law, or if you can find an even number that is not the sum of two primes, your name will appear in headlines in the mathematical world.

13-3. Distribution Of Primes

Look at the list of prime numbers at the end of this chapter. This list has some interesting properties. For instance, we have already noticed that 2 is the only even prime number. This means that once we are beyond 2 all the prime numbers are odd. Thus the difference between any two consecutive prime numbers of the sequence is always even. (Why?) Find a pair of consecutive primes whose difference is 4. Do the same for each of the differences 2, 6, 8, 10, 12, 14. Is there a difference larger than 14 in the part of the sequence from 2 to 307? As a matter of fact, if you examine the sequence of primes as far as 500, there is still no gap greater than 14. On the basis of this evidence you might guess that there are no gaps greater than a certain number, perhaps 14.

This conjecture, however, is false. In fact as far back as about 300 B.C., Euclid proved that there are gaps as large as we please in the sequence of primes. For instance, let N be the result of multiplying all the counting numbers from 2 to 101.

$$N = 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \ldots 101.$$

You had better not take time to multiply out his product, since if you express N in the usual decimal notation, it would be a numeral of about 160 digits!!

The number N is divisible by each counting number from 2 to 101, since N has each of those numbers as factors. Now look at the number N + 2. We can always write,

$$N = 2 \cdot (3 \cdot 4 \cdot \ldots \cdot 101) = 2S,$$

where S stands for the product of all the factors of N except 2. Then,

$$N + 2 = 2S + 2 = 2(S + 1),$$

where the last statement is obtained by using the distributive property. Then N + 2 is divisible by 2 and therefore is not a prime.

Now consider the number N + 3. By use of the commutative and associative properties we can write

$$N = 3(2 \cdot 4 \cdot 5 \cdot 6 \ldots 101) = 3T$$

where T stands for the number in the parenthesis. Thus

$$N + 3 = 3T + 3 = 3(T + 1).$$

Here again we have used the distributive property. This shows that N + 3 has a factor 3 and so is not prime. In the same way we may show that N + 4 has a factor 4, N + 5 a factor 5, and so on, until N + 101 has a factor of 101. Thus none of the hundred numbers

$$N + 2,\ N + 3,\ N + 4,\ \ldots,\ N + 101$$

is prime so there is a gap of at least 100 between successive primes. It should be clear how this method can be used to show that there are gaps as large as we wish.

Now that we have looked for large gaps, let us consider small ones. What is the shortest possible gap between two primes greater than 2? Two odd primes whose difference is 2 are called twin primes, or simply twins. A prime which belongs to such a pair is called a twin. To get an idea of how frequently twin primes occur, examine the following table comparing the number of twins less than some number n with the total number of primes less than n.

n	number of primes < n	number of twins < n	$\frac{\text{number of twins} < n}{\text{number of primes} < n}$
20	8	7	$\frac{7}{8} \approx .88$
40	12	9	$\frac{9}{12} = .75$
60	17	12	$\frac{12}{17} \approx .71$
80	22	15	$\frac{15}{22} \approx .68$
100	25	15	$\frac{15}{25} = .60$

Verify this table by checking it with your list of primes. Does the table seem to indicate that the twins are becoming rarer in comparison with the number of primes? Test this conjecture further by completing the table in Problem 1 of the exercises below. On the basis of this evidence, would you conjecture that the twins are rare in comparison with the number of primes? As a matter of fact, it is known that in a certain sense the twins do become rare as we continue the sequence of primes, although we will not try to pursue this topic in detail here. How rare do they become? For example, do they stop altogether after a certain point? Here again we come to some of the mathematics nobody knows. The answer to this question is not yet known.

As you have seen, many questions and conjectures suggest themselves with respect to the counting numbers and a substantial number of these have not yet been settled. The branch of mathematics dealing with these subjects is called Number Theory. If you wish to go more deeply into this you may read the two booklets on this subject, called "Essays on Number Theory," published by SMSG. Further references may be found there and in "Study Guide in Number Theory" published by SMSG.

A few other ideas and conjectures in the area of Number Theory are suggested in the exercises below.

Exercises 13-3a

1. Complete the table below.

n	number of primes < n	number of twins < n	$\dfrac{\text{number of twins} < n}{\text{number of primes} < n}$
100	25	15	$\frac{15}{25} = .60$
200	46	29	$\frac{29}{46} \approx .63$
300	62	37	
400	78	41	
500	95	47	
600			
700			
800			
900			
1000			

2. For each row in the table of Problem 1 calculate the product of the numbers in the 1st and 3rd columns divided by the square of the number in the second column. For the first row you calculate $\dfrac{(100)(15)}{(25)^2}$, for the second row $\dfrac{(200)(29)}{46^2}$, and so on. Do your results suggest a conjecture?

3. Complete the following table showing the number of primes between successive perfect squares.

Interval	Number of Primes
1 - 4	2
4 - 9	2
9 - 16	2
16 - 25	
25 - 36	
36 - 49	
continue this to	
225 - 256	

For example, there are 2 primes between 9 and 16 (what are they?), so we have written a 2 in the third row of the second column. Does this table suggest a conjecture to you? Actually it is not known whether there is always one prime between any two consecutive squares. Here again is a bit of the mathematics nobody knows.

4. You may classify the odd prime numbers according to their remainders when they are divided by 4. For example, 5, 13, 17 have remainders of 1 while 3, 7, 11 have remainders of 3. If the remainder is 1, the number can be written in the form $4k + 1$ for some k. If the remainder is 3, it can be written in the form $4k + 3$. List your results by completing the following table.

n	number of primes of form $4k + 1$ which are less than n	number of primes of form $4k + 3$ which are less than n
10	1	2
20	3	4
30	4	5
40		
50		
60		
70		
80		
90		
100		

What conjecture does this table suggest to you? Test your conjecture by continuing the table to 200. Does this prove the conjecture is true?

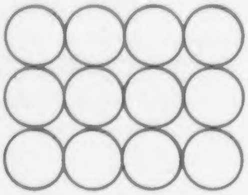
*5. If you examine the list of primes you will notice a case in which three consecutive odd numbers are primes. What are the primes? Do you find any other example of this? Would you guess that this never happens again? Try to show that this conjecture is correct by showing that for three consecutive odd numbers one of them is always divisible by 3. The number divisible by 3 cannot be prime except when it equals 3. Thus three consecutive odd primes occur only in the case of 3, 5, 7, which has already been found.

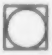
13-4. Problems on Spheres

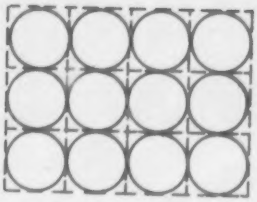
Let us now turn to a problem in a different field. Everyone who has ever held a bag under the nozzle of a coffee grinding machine at the store knows that when the grinding is finished the bag seems very full, but that on shaking it the coffee grounds settle considerably. The vibration causes the particles of coffee to pack together more closely and hence to fill less volume. This experience suggests the following packing problem, which has several important applications. Suppose you have a large number of perfectly spherical marbles which you propose to pack into a barrel. How should you pack the marbles so that you get in as many as possible? We imagine here that the barrel is so large in comparison to the marbles that its exact dimensions do not matter and we are really just asking, "How do you pack the marbles so you have the greatest number of marbles per cubic foot?"

Perhaps we should begin with the corresponding plane problem. If you have a number of identical circular disks, say pennies, how do you arrange them on the plane without overlapping so as to get as many as possible in a given large plane region, say a large rectangle? For convenience, suppose the diameter of the pennies is taken as a unit of distance.

One way of arranging the pennies would be as shown below,

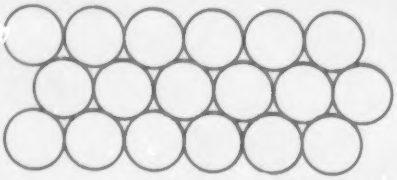


in which each penny except those on the edge touches four others. This really amounts to thinking of each circle as inscribed in a square one unit on a side, like this , and then fitting the squares together so that their interiors fill out the plane region.

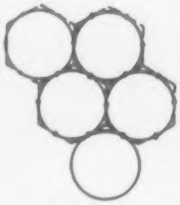


What is the area of the interior of each square? What is the area of each disk? Show that the ratio of the area of the disk to that of the square is $\frac{\pi}{4}$. Show that this is approximately .785, so that each disk covers about 78.5% of the corresponding square region. Since squares fit together without overlap to cover the plane, this means that the disks in this arrangement cover about 78.5% of the plane region, or about 21.5% is left uncovered.

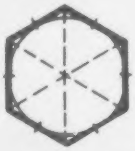
Can you think of a better way of fitting the pennies together? Probably the following arrangement has already occurred to you as a good possibility.



In this arrangement each inner disk touches how many others? It is as though each disk had been inscribed in a regular hexagon and the hexagons fitted together as shown by the dotted lines in the following figure.



Each of the hexagons may be thought of as made up of six equilateral triangles as in the figure below. The altitude of each such triangle has a length equal to the radius which is $\frac{1}{2}$. Show that the interior of each hexagon has an area of $\frac{\sqrt{3}}{2}$ square units. (The length of a side opposite a 30° angle in a right triangle is equal to the length of the side adjacent to the angle, divided by $\sqrt{3}$.)



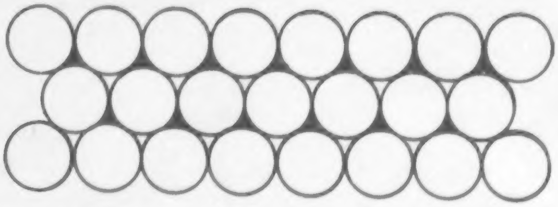
Since the area of the disk is $\frac{\pi}{4}$, the ratio of the area of the disk to that of the interior of the hexagon is $\frac{\pi}{4} \div \frac{\sqrt{3}}{2} =$

$\frac{\frac{\pi}{4}}{\frac{\sqrt{3}}{2}} \times \frac{4}{4} = \frac{\pi}{2\sqrt{3}} = \frac{\pi}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\pi\sqrt{3}}{6}$. Show that this is approximately .907. This shows that the disk covers 90.7% of the hexagon. Since we have observed that hexagons fit together to cover a plane region without overlapping, in this arrangement some 90.7% of the plane region is covered by disks. This is far better than our previous arrangement.

Can you think of other arrangements of the pennies which might be better than this? Actually the arrangement above can be shown to be the best one, though we will not try to give a proof of this fact.

Now let us return to the problem of packing the marbles in the barrel. Do you have a conjecture about the best way to pack them? Experiment with some marbles before you go on and see if you can make a conjecture as to the most efficient packing.

One possible procedure would be to start by putting a layer of marbles with their centers all in a plane parallel to the bottom--that is, a layer on the bottom of the box or barrel or whatever we are filling. From the top, they will look just like a covering of a plane region with circles, and in the light of our discussion about circles it seems that we should arrange our spheres in the same way, as shown below.



Now it seems plausible that we should try to make a second layer of marbles. Do you think it would be a good idea to place this layer of marbles such that each marble is directly above one in the first layer? No, it appears that we would get a better packing by trying to place the marbles of the second layer over the "pockets" or "holes" in the first layer. Actually there is not room to put a marble above each hole, but we can place a layer above half of the holes, say those shaded in the figure above. Then a third layer can be placed on the second, covering half of the "holes" of the second layer. This can be done in two ways, depending on which set of "holes" we choose to fill. The spheres of the third layer may be exactly above those of the first or may be above the unshaded "holes" of the first layer.

This seems to be a reasonable conjecture about the best possible packing. It can be shown that for this method of packing the ratio of the volume of the marbles to that of the region is $\frac{\pi}{3\sqrt{2}} \approx 0.7405$, so this packing fills about 74% of the space with spheres. No one knows whether or not this is really the best packing. The best result known so far was obtained by a British mathematician, Rankin, in 1947, who showed that there is no packing in which the spheres fill more than 82.8% of the volume of space.

If you feel certain that you know the correct answer to this problem, even if you cannot prove it, consider the problem of packing a mixture of marbles of different sizes. For example, you may have some marbles with a diameter of two inches and others with a diameter of one inch. Suppose you want to pack a barrel with three times as many one-inch as two-inch marbles. How do you get in as many as possible? So far, nobody has even a good idea as to how to attack this problem.

The problems we have just dealt with are theoretical and the mathematician is interested in them for themselves, but there are applications here also. In insulating material,one is interested in having airspace in the form of small "pockets" or air which are not large enough to permit circulation. One way to simplify such problems is to consider the packing of small spheres between two layers of hard material. Sometimes the surface area of the spheres must be taken into account, as well as the physical properties of the materials themselves. Similar problems occur in the design and testing of plastics.

Exercises 13-4a

1. Take the diameter of a penny as a unit of length. Draw a square 8 units on a side (area = 64 square units). Try to arrange pennies to fit as many as possible into the region. How many did you get? Find the ratio of the total area of the circles to the total area of the region. How does this compare with the theoretical result of 0.907 obtained above? Take a square region 14 units on a side (area = 196 square units) and try the experiment again. As the region gets larger the approximation to the theoretical result should improve.

2. Suppose a set of disks each have diameter 1 and we seek to pack as many as possible in a region whose area is 1000 square units. Using the result that at most 90.7% of the region can be covered, what is the largest number of disks that could be fitted into the region?

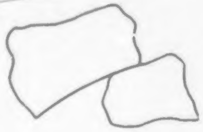
3. Take a convenient box like a chalk box and a collection of marbles all of the same size. See how many you can pack into the box. Count the number. Find the ratio of the total volume of the marbles to the total volume of the box and compare it with the 0.74 figure noted above.

4. When you use a packing of spheres such as was discussed in this section, each interior sphere in the packing touches how many other spheres?

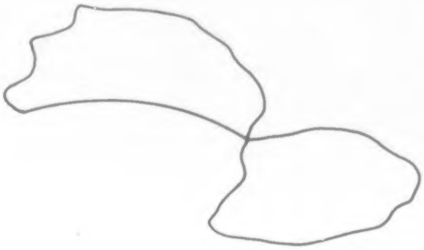
13-5. A Problem on Coloring Maps

Suppose that we have a map and a box of crayons. We wish to color the map. In order to distinguish clearly the different countries, when two countries border each other

along an arc like this,
we shall color them differently.

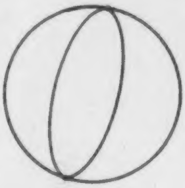


If they have a single boundary point in common, we allow them to have the same color.

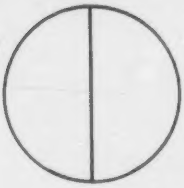


In coloring a map we agree also to color the outside region; you can imagine this is the surrounding ocean if you wish.

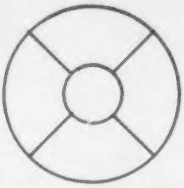
It is a reasonable question to ask how many different-colored crayons will be needed to color a map. What would be the smallest number of colors needed for the following map?



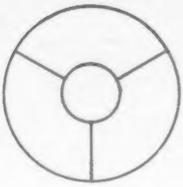
What about this map?



Can you color the following map in less than four colors?



How about this map?

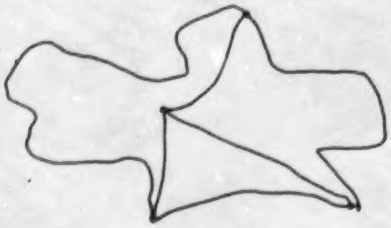


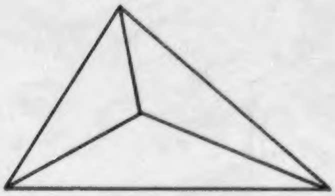
Why could you not get along with less than four colors here? Notice the inner country and the three in a ring about it. Is it true that each of these countries has a boundary arc in common with each of the others? Could any two of them be colored the same color? Does this show why at least four colors are necessary?

Can you think of a map that requires five colors? Of course if you could draw a map in which there were five countries, each of which bordered all of the other four this would take five colors, just as the map above took four colors. Try to find such a map. Were you successful? Actually it is not too hard to guess that it is not possible to have five regions each bounding the other four. This does not show, however, that there are not some maps requiring five colors. Do you think there are maps needing five colors?

In a sense your guess about the answer to this question is as good as anyone's, since the answer is not yet known. There is a legend (of very doubtful authenticity) that a mapmaker proposed this question about 100 years ago to the British mathematician, Cayley. In any case, the map coloring problem is now one of the classic unsolved problems of mathematics. In 1897 the British mathematician, Heawood, showed that any map can be colored in five colors, so that in any case no more than five colors are necessary. In 1941 the American mathematician, Franklin, proved that every map with less than 38 countries could be colored in four colors. So if you look for a map which really needs five colors it will have to be fairly complicated since it must have at least 38 regions.

There is one very interesting feature of the problem. If we consider any map, the number of colors needed clearly does not depend on the particular shape of the bounding arcs. In fact, if we think of the map as drawn on a rubber sheet, then the sheet may be stretched and twisted continuously in all possible ways without changing the coloring problem. Thus, for example, the two maps below may be considered the same since we can always distort one into the other.



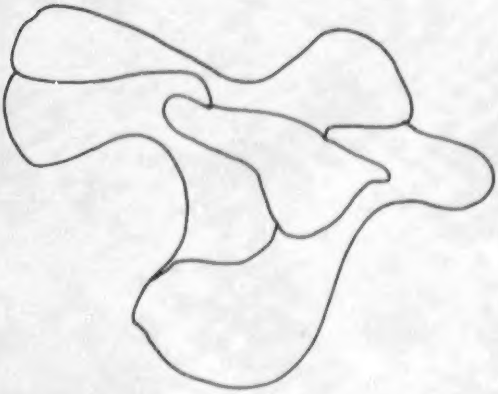


This is what is meant by saying that the problem is a topological one. That is, the solution of the problem is unchanged by any continuous distortion of the figure. One convenient result of this is that we may always imagine that a map has been distorted in such a way that all the arcs have been made into unions of line segments. This means that it is really enough to consider maps in which the regions are polygons (triangles, quadrilaterals, pentagons, and so on) as is done in most of the examples below.

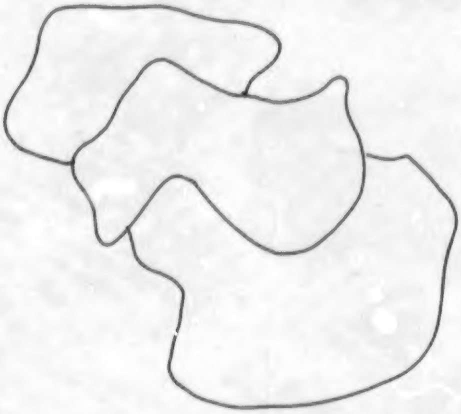
Exercises 13-5a

1. Draw a polygonal map which is equivalent to each of the following--that is, into which each of the following can be distorted.

(a)

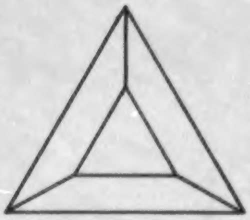


(b)

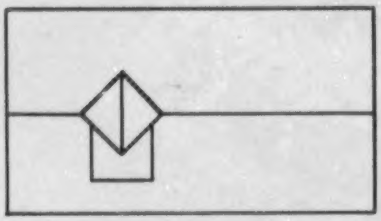


2. Color each of the following maps in as few colors as possible.

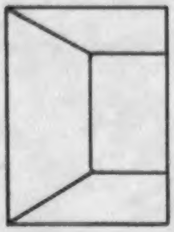
(a)



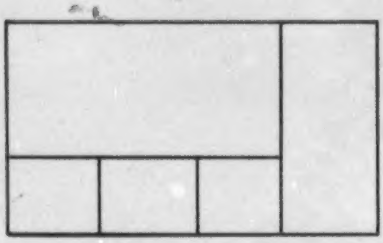
(b)



(c)



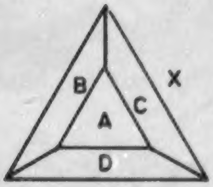
(d)



3. Suppose we have two islands surrounded by ocean. One island is divided into countries as shown in the figure for problem 2a and the other as in the figure for problem 2b above. Color this map in four colors.

4. Suppose you have an ocean with two islands, each divided into countries, and suppose you know how to color each island and the surrounding ocean in four colors. Show how you can color the combined map in four colors.

An interesting related question in map coloring is as follows. Suppose you have a given fixed number of colors in your crayon box. In how many different ways can you color a given map? Of course, if you don't have enough colors, the answer is zero. As an illustration look at the map of the figure for problem 2a and suppose there are five colors in your box. The color schemes for this map are shown below where regions marked A, B, C, D



must all be different colors, while the region marked X may be any color except those used for B, C, and D. Why are these the only schemes? Notice that X may be the same color as A or it may be a color that has not been used at all before.

In this scheme you have five choices for the color to use for the region marked A. Once you have chosen a color for A, how many choices do you have for a color for the region marked B? In how many different ways can you color the regions marked A and B? Clearly this is $5 \times 4 = 20$ ways. For example, if the five given colors are red, green, yellow, blue, and white, the twenty possible pairs of colors for regions marked A and B are:

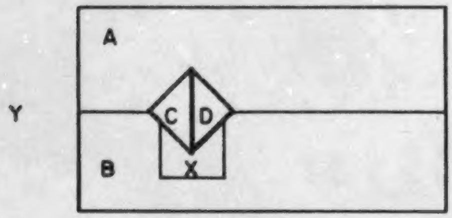
red-green	red-yellow	red-blue	red-white
green-red	green-yellow	green-blue	green-white
yellow-red	yellow-green	yellow-blue	yellow-white
blue-red	blue-green	blue-yellow	blue-white
white-red	white-green	white-yellow	white-blue

If you are doubtful about this reasoning, look back at Chapter 7, where you did problems of this kind. For each of these 5×4 choices of colors you still have 3 choices for a color for the region marked C and then 2 choices for a color for the region marked D. Finally, since region X can be any color except those used for B, C, D, there are $5 - 3$ or 2 choices for a color for X. The total number of ways of coloring map 2(a) in at most five colors is thus $5 \cdot 4 \cdot 3 \cdot 2 \cdot 2 = 240$ ways.

The following problems, which are a continuation of Exercises 13-5a, deal with the number of ways of coloring a map.

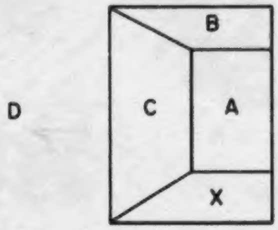
Exercises 13-5a (continued)

5. (a) Show that the different possible color schemes for map 2(b) can be described as follows:



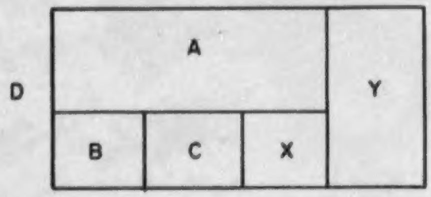
where regions marked A, B, C, D must be distinct colors; region X may be taken as any color except those of B, C, D; and finally , region Y as any color other than the colors of A or B.

(b) Show that the different possible color schemes for map 2(c) can be described as follows:



where colors for regions A, B, C, D must be distinct, but X can then have any color except those of A, C, D.

(c) Show that the different possible color schemes for map 2(d) can be described as follows:



where regions marked A, B, C, D must be different colors; X can be any color other than the colors of A, C, D; and finally, Y can be chosen as anything other than the colors of A, X, D.

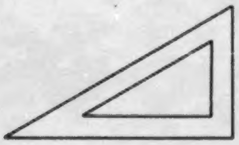
6. Use the results of Problem 5 and the method shown above to verify and complete the following table showing the number of ways of coloring the different maps in n colors for certain values of n.

n	Map 2(a)	Map 2(b)	Map 2(c)	Map 2(d)
3	0	0	0	
4	24	48	24	
5	240	720		480
6	1080	4320		

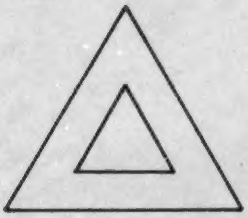
7. In map 2(a), if we have n colors to use, there are n choices for a color for region A, then $(n - 1)$ choices of color for B, $(n - 2)$ for C, $(n - 3)$ for D and finally $(n - 3)$ for X, since X was any color except those of B, C, D. The total number of ways of coloring 2(a) with n colors is therefore $n(n - 1)(n - 2)(n - 3)^2$. Verify that this does give the correct results for $n = 3, 4, 5, 6$ by comparing with Problem 6.

8. Use the method of the last problem to find formulas for the number of ways of coloring each of the maps 2(b), 2(c), and 2(d) in n colors.

We noticed earlier that as far as the map coloring problem is concerned any map can be replaced by a polygonal map--that is, one in which all the boundaries are unions of line segments. Thus, if every polygonal map can be colored in a certain number of colors, then every map can be colored with the same number of colors. We also mentioned that a mathematician named Heawood, actually proved that any polygonal map (and thus any map) can be colored in five colors. Now it is clear that to prove a result that applies to all maps it is necessary to know some properties that hold for all maps. While we will not try to show Heawood's proof, it will be interesting to discover a general law about polygonal maps which was one of Heawood's main tools. This law applies to all polygonal maps without islands--that is, maps for which the boundaries make up one connected piece. For example the given figure has an island and is not considered.



The property we are going to state was discovered by Euler about 200 years ago. Actually, it was discovered a hundred years earlier by the great French mathematician Descartes (pronounced "day-cart"), but so few people read Descartes' work that when Euler discovered it again everyone thought it was new. Somewhat illogically Euler always gets credit for it, as it is called Euler's formula.



Look at Figure 2(a) of Exercises 13-5a. How many vertices do you see in this map? How many countries or regions are there? (Do not forget to count the outside region.) How many edges or segments are there in the boundary? Do your answers agree with the figures in the first column in the chart below? The letters V, C, E are used to stand for the numbers of vertices, of countries or regions, and of edges.

	Map 2(a)	Map 2(b)	Map 2(c)	Map 2(d)
V	6			
C	5			
E	9			

Exercises 13-5b

1. Examine maps 2(b), 2(c) and 2(d) of Exercises 13-5a and complete the chart above.

2. Make at least five polygonal maps without islands. Count the vertices, edges, and regions and tabulate them in a table like the one above.

3. Examine the tables you have made and see if you can find a relation between V, C, E which holds for all the maps you have drawn. Can you make a conjecture as to a relation that holds for all polygonal maps without islands?

4. Make five more polygonal maps without islands and test your conjecture on these.

5. Compare your results with those of your classmates? What relation between V, C, E do you conjecture to be true for all maps of the kind discussed.

Notice that the discussion above does not prove Euler's formula. The proof would still have to be given, but it helps to have at least a good guess as to what is to be proved.

In Section 10-8, Euler's formula was discussed for simple surfaces. There we used F for the number of faces in place of C for the number of countries. The treatment in this chapter is for use with plane polygonal paths, while in Chapter 10 we were studying surfaces. In many respects these topics are equivalent.

13-6. The Travelling Salesman

The first problems discussed in this chapter, namely those on prime numbers, could be classed as problems in pure mathematics. Yet we saw that the properties of prime numbers are often used in present-day applications to computing. Also, the sphere-packing problem and the map-coloring problem are not without usefulness and applications.

In this section we consider a problem which sounds very practical and is tremendously important in application. It is called "the travelling salesman problem." Although it is really not used by travelling salesmen, the essence of the problem can very easily be described in terms of salesmen who travel, and mathematicians have come to think of it under this name. In actual applications, "salesman" could be replaced by "plane" and "city" by "base"; other replacements might be "truck" and "factory," respectively.

Let us suppose you are a travelling salesman with home office in a city A. It is your duty to make a monthly visit to customers in cities B, C, D, and return to the home office. Your problem is to arrange this trip as efficiently as possible--that is, to make the distance travelled as short as possible. On the face of it this is a very simple problem. All you need to do is look at each of the possible routes, add up the distances and see which is shortest. Let us see how many routes there are. How many choices are there for the first stop after leaving home? Clearly 3, since there are 3 cities to visit. Now that you have visited this first city, how many choices are there for the next stop? Can you find the total number of possible routes? You should find $3 \cdot 2 \cdot 1 = 6$ routes to consider. In fact, we can write them down like this, listing in order the places visited.

ABCDA	ADCBA
ABDCA	ACDBA
ACBDA	ADBCA

As a matter of fact the problem is not even as bad as this, for the two routes listed in each row are merely the same route in opposite directions and clearly have the same length; so it is only necessary to find the total lengths of three routes and pick the shortest one. Problem 1 in Exercises 13-6a is an example of this.

But suppose your sales territory expands and you are assigned 8 cities to visit. Now how many possible routes are there? A similar argument this time shows that there are $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ routes or 40,320 possible routes. Even if you divide this by 2, since you actually have counted a route going both ways, there are still 20,160 different routes to check, a formidable task.

In 1954, Dantzig, Fulkerson, and Johnson considered the travelling salesman problem for a man based at Washington, D.C., who was to visit the 48 state capitals (Hawaii and Alaska hadn't yet been admitted). The number of possible routes is, of course, by the same reasoning

$$48 \cdot 47 \cdot 46 \cdot \ldots \cdot 3 \cdot 2 \cdot 1.$$

This is a number which, expressed in the usual base 10 notation has 61 digits, a computational problem that is beyond the capacity of even the best high speed computers.

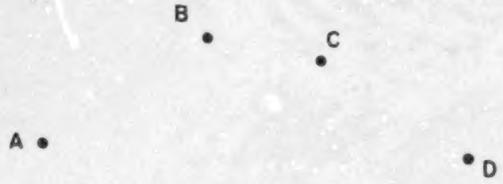
The reason for the difficulty is clear. In solving this problem there are a definite number of cases to consider, but the number is so large that it is hopeless to try to deal with the cases one by one except in the simplest situations. The method used by Dantzig, Fulkerson, and Johnson involved a great deal of work with high speed computors. A good general method for solving problems of this kind in a reasonable amount of time has not yet been found.

Perhaps in working with this problem we should not be so ambitious. Maybe we should not insist on the very best result, but should look for a method with a high probability of coming close to the best answer. If you have any good ideas on the subject, there are certainly a lot of industries and government agencies who would be very interested.

Problems of this kind are not only interesting in themselves, but a number of practical problems involving transportation, scheduling of factory operations, and designing communication networks lead to such questions. A few more examples are suggested in the problems below. A very readable book on somewhat similar problems is one by Williams called "The Compleat Strategyst." You might enjoy examining this.

Exercises 13-6

1. This diagram is intended to show four cities.



   Suppose the distances in miles between different pairs of cities are d(AB) = 200, d(AC) = 300, d(AD) = 400, d(BC) = 150, d(BD) = 250, d(CD) = 180. Solve the travelling salesman problem for this case by finding the shortest route starting and ending at A, and passing through all the cities.

2. In the travelling salesman problem, it ought to be more important to make the travelling time as short as possible rather than the actual distance. Moreover, because of plane or bus connections, the time required to get from A to B may not be the same as from B to A. Suppose in Question 1 the travelling times in hours are t(AB) = 3, t(BA) = 2, t(AC) = 4, t(CA) = 4, t(AD) = 5, t(DA) = 7, t(BC) = 1, t(CB) = 2, t(BD) = 2, t(DB) = 4, t(CD) = 3, t(DC) = 1. Find the route for the salesman which will take the least time.

3. If there are 10 cities, how many routes begin and end at a given city?

4. (a) In the square below, how many sets of three numbers can be chosen with no two from the same row or column?

9	3	6
4	8	2
5	7	1

   (b) Which one or ones of these choices gives the largest sum for the three numbers?

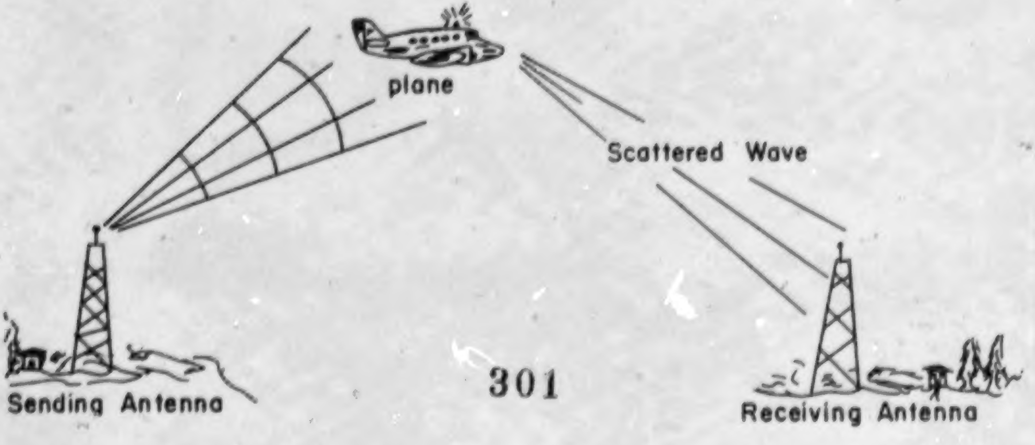
5. Solve the problem corresponding to Problem 4 for the square below.

1	9	6	14
7	15	2	10
3	11	8	16
5	13	4	12

13-7. Applied Physical Problems

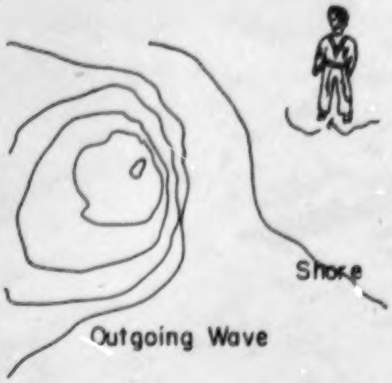
Some of the most interesting as well as some of the most important of the unsolved mathematical problems are related to physical problems like propulsion of rockets, motion of airplanes, and behavior of electromagnetic waves. In order to really appreciate most of these problems in science and engineering you need to know some topics both in mathematics and science which you have not yet studied. However, perhaps we can give a brief hint at the meaning of one or two of these questions.

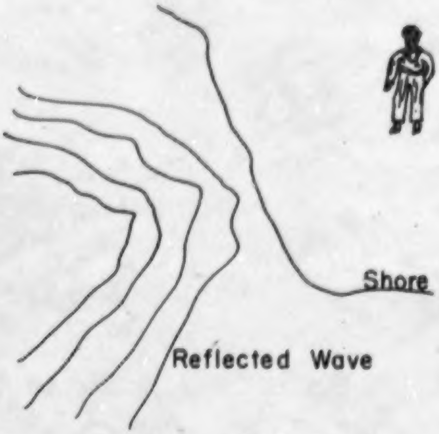
For example, consider the basic problem of radar. We send out from one antenna an electromagnetic wave. This is reflected from some obstacle and the scattered wave is picked up on a receiving antenna; this produces a "blip" on the radar screen.



If the shape of the obstacle, here an airplane, is known, mathematicians have learned fairly well how to figure out what the scattered wave is like. Unfortunately, we do not know how to examine the scattered wave and to decide what the obstacle was like. Thus, we cannot tell from the appearance of the "blip" on the screen what sort of an object produced it. You can see why it would be very useful if we could do this.

The problem is somewhat similar to the following one. Suppose you are standing by the seashore in a rather heavy fog. Somewhere out there in the fog is some object, perhaps a ship, but it is hidden by the fog. You pick up a big rock and throw it in the water creating waves. In a short time you see these waves coming back having been reflected off the unseen obstacle. You would like to be able to look at those returning waves and decide whether the object out in the water is a rowboat, or a garbage scow, or the Queen Mary.





Other interesting problems are concerned with the effect of the flow of air around a plane, and how to design a plane with useful flying characteristics.

Still other examples of unsolved questions in applied mathematics are:

1. Is the solar system stable--that is, will the planets indefinitely move around the sun in approximately their present orbits or will they ultimately disperse into space or come into the sun? Or, more modestly, what about orbiting satellites?

2. Which species of animals will survive and which will die out? Can we develop a quantitative theory of biological evolution?

3. How can we design a computer to do a specific job using the smallest possible number of components of given types? And can a process for answering this question be put in the form of a program of instructions for existing computers, so we could use present computers to design better ones?

4. How does a given distribution of transportation facilities--highways, railroads, airports, etc.--affect the growth and distribution of population and industry in the future?

13-8. Conclusion

In 1900 an international congress of mathematicians met in Paris. One of the greatest mathematicians of our century, a German, named David Hilbert (1860-1942), gave a speech in which he listed 23 unsolved problems for which he considered the solutions to be most important for the progress of mathematics in the following century. Of course, during the last sixty years a large number of other major problems have been formulated. Still, one of the quickest ways even now for a young mathematician to make a reputation is to solve one of Hilbert's problems. Understanding many of these problems requires advanced mathematics, but one, for example, concerns some readily understood properties of decimals. We know that any real number has a decimal representation. The number $\sqrt{2}$ in particular has such a representation,

$$\sqrt{2} = 1.4142\ldots ,$$

and we know how to calculate any number of digits in this expansion. We know that the decimal expansion doesn't end and is not periodic (See Chapter 6), but beyond that we know almost nothing about this number. For instance are there infinitely many 1's or are there seven consecutive 7's? No one knows, and so far there seems to be no method of finding out.

At present, about half of Hilbert's problems have been solved, and the century still has almost 40 years to go. Who will solve these problems of Hilbert, and the new problems that have come to light since Hilbert, and the problems that haven't been asked yet but will be? No one can say, of course, but it is quite probable that most of them will be solved by comparatively young people. Most of the great discoveries in mathematics have been made by people in their 20's and early 30's. If a person has mathematical talent it usually shows up early. The average age for getting the Ph.D in mathematics is lower than in almost any other field. So, in all probability the problems mentioned in this chapter will not be solved by old gray-bearded professors (or do professors have beards any more?), but by youngsters not much older than you.

But mathematics is not just for those who make discoveries in the subject for the love of it. It is rapidly increasing in importance for such fields as medicine, biology, psychology, sociology, and economics, as well as in new fields of physical science. As machines take over many of the occupations of unskilled labor, the demand for trained personnel increases, and more and more such training is connected with mathematics. Furthermore, the American citizen who goes to the polls today must have a more extensive knowledge of mathematics than the American citizen of yesterday, so that he may be able to make the important decisions required of him in a democracy such as ours. It is vital that he must, on the one hand, recognize the role and the importance of scientists and mathematicians, but on the other hand he must be knowledgeable enough to understand and assess, to a certain extent, their claims.

So, whether or not you are among those who advance mathematics for its own sake and your love of it, or are able to apply it to other fields, or use it as a source of puzzle problems in your recreation, or through your knowledge be aware of its role in modern civilization, it will affect your life. And, who knows, you may affect mathematics!

List of Prime Numbers less than 1000

	0	1	2	3	4	5	6	7	8	9
0		2,	3,	5,	7,	11,	13,	17,	18,	23,
1	29,	31,	37,	41,	43,	47,	53,	59,	61,	67,
2	71,	73,	79,	83,	89,	97,	101,	103,	107,	109,
3	113,	127,	131,	137,	139,	149,	151,	157,	163,	167,
4	173,	179,	181,	191,	193,	197,	199,	211,	223,	227,
5	229,	233,	239,	241,	251,	257,	263,	269,	271,	277,
6	281,	283,	293,	307,	311,	313,	317,	331,	337,	347,
7	349,	353,	359,	367,	373,	379,	383,	389,	397,	401,
8	409,	419,	421,	431,	433,	439,	443,	449,	457,	461,
9	463,	467,	479,	487,	491,	499,	503,	509,	521,	523,
10	541,	547,	557,	563,	569,	571,	577,	587,	593,	599,
11	601,	607,	613,	617,	619,	631,	641,	643,	647,	653,
12	659,	661,	673,	677,	683,	691,	701,	709,	719,	727,
13	733,	739,	743,	751,	757,	761,	769,	773,	787,	797,
14	809,	811,	821,	823,	827,	829,	839,	853,	857,	859,
15	863,	877,	881,	883,	887,	907,	911,	919,	929,	937,
16	941,	947,	953,	967,	971,	977,	983,	991,	997,	1009.

Explanation of table of primes: This table is arranged so that it is easy to pick out, for instance, the 47th prime number. To do this, use the row with the number 4 at the left and 7 at the top. Then you see that the 47th prime number is 211. This also can be used the other way around. Since the number 691 occurs in the row labeled 12 and the column labeled 5, the number 691 is the 125th prime.

SUBJECT CLASSIFICATION*

FOUNDATIONS, THEORY OF SETS, LOGIC

1. Philosophy of mathematics and physics
2. Foundations of mathematics, axiomatics
3. Intuitionism
4. Set theory
5. Problem of the continuum
6. Transfinite numbers
7. Relations
8. Syntax, semantics, formal methods in general, recursive functions
9. Logical calculi, many-valued, modal
10. Applications of logic

ALGEBRA

Universal Algebra, Miscellaneous Algebraic Systems

Combinatorial Analysis

Algebra: Elementary

Linear Algebra
1. Forms and transformations
2. Matrices
3. Inequalities for matrices
4. Eigenvalues and eigenvectors
5. Multilinear forms, alternating forms, Grassmann algebra
6. Linear equations, matrix inversion, determinants

Polynomials
1. Algebraic equations, roots
2. Symmetric functions
3. Reducibility

Partial Order, Lattices
1. Lattices
2. Boolean rings and algebras

Fields, Rings
1. Fields
2. Finite fields
3. Galois theory
4. Valuations
5. Rings
6. Ideals

Algebras
1. Associative algebras
2. Non-associative algebras
3. Lie algebras
4. Differential algebra

Groups and Generalizations
1. Group theoretic constructions, free groups, extensions
2. Abelian groups
3. Nilpotent and solvable groups
4. Finite groups
5. Ordered groups
6. Matrix groups, representation, characters
7. Semigroups
8. Other generalizations of groups
9. Applications

Homological Algebra

THEORY OF NUMBERS

Theory of Numbers: General
1. Elementary number theory
2. Magic squares
3. Congruences
4. Diophantine equations
5. Representation problems
6. Divisibility and factorization
7. Power residues and reciprocity laws
8. Forms
9. Fermat's last theorem
10. Number-theoretical functions
11. p-adic numbers

*1958 Index of Mathematical Reviews.

Theory of Numbers: Analytic

1. Analytic theory in number fields and fields of functions
2. Analytic tools (zeta-function, L-functions, Dirichlet series)
3. Distribution of primes
4. Additive number theory, partitions
5. Equidistribution, statistical number theory
6. Irrationality and transcendence

Algebraic Numbers

1. Class fields

Geometry of Numbers, Diophantine Approximations

1. Diophantine approximations
2. Geometry of numbers

ANALYSIS

Functions of Real Variables

1. One real variable
2. Several variables
3. Calculus, mean-value theorems, inequalities
4. Differentiation and tangents
5. Non-differentiable functions; generalized derivatives
6. Representation of functions by integrals
7. Quasi-analytic functions

Measure, Integration

1. Measure theory
2. Measure-preserving transformations, ergodic theorems
3. Riemann integrals
4. Stieltjes and Lebesgue integrals
5. Denjoy, Perron integrals
6. Abstract integration theory, somas
7. Area, length
8. Product integrals
9. Abstract theory of probability

Functions of Complex Variables

1. Foundations
2. Quasi-conformal functions
3. Generalizations
4. Power series
5. Zeros
6. Singularities
7. Analytic continuation, overconvergence
8. Cauchy integral
9. Maximum principles, Schwarz lemma, Phragmén-Lindelöf theorem
10. Conformal mapping, general
11. Conformal mapping, special problems
12. Riemann surfaces and functions on them
13. Entire functions
14. Meromorphic functions
15. Distribution of values, Nevanlinna theory
16. Behavior on the boundary
17. Univalent and p-valent functions
18. Bounded functions, functions with positive real part, etc.
19. Iteration
20. Normal families
21. Expansion in series of polynomials and special functions
22. Continued fractions
23. Complex interpolation and approximation
24. Functions of several complex variables

Harmonic Functions, Convex Functions

1. Harmonic functions, potential theory
2. Subharmonic functions
3. Biharmonic and polyharmonic functions
4. Generalized potentials, capacity
5. Harmonic forms and integrals
6. Convex functions

Special Functions

1. Polynomials as functions, orthogonal polynomials
2. Exponential and trigonometric functions
3. Elliptic functions and integrals, theta functions, complex multiplications
4. Automorphic functions, modular functions
5. Bessel functions
6. Legendre functions, spherical harmonics
7. Lamé, Mathieu functions
8. Hypergeometric functions and generalizations
9. Functions defined by definite integrals, differential and integral equations, infinite series

Sequences, Series, Summability

1. Special sequences and series, moments
2. Power and factorial series
3. Dirichlet series
4. Operations on series and sequences
5. Convergence and summability
6. Tauberian theorems

Approximations and Expansions

1. Interpolation: general theory
2. Approximations and expansions, general theory
3. Orthogonal systems, expansions
4. Closure
5. Degree of approximations, best approximation
6. Asymptotic approximations and expansions

Trigonometric Series and Integrals

1. Trigonometric polynomials, Fourier series
2. Trigonometric interpolation
3. Fourier coefficients, degree of approximation
4. Convergence, summability
5. Absolute convergence
6. Double and multiple series
7. Fourier integrals
8. Almost periodic functions

Integral Transforms

1. Inversion formulas, self-reciprocal functions
2. Laplace and Fourier transforms
3. Other transforms, Hilbert, Mellin, Hankel

Differential Equations: Ordinary

1. Existence and uniqueness
2. Approximation of solutions
3. Asymptotic behavior of solutions
4. Singularities of solutions
5. Linear equations: second order
6. Linear equations: other than second order
7. Systems of linear equations, matrix differential equations
8. Stability of solutions
9. Periodicity, oscillations
10. Boundary value problems, spectra, expansions in eigen-functions
11. Dynamical systems, topological properties
12. Special types

Differential equations: Partial

1. Existence uniqueness, stability
2. Total equations, Pfaff's problem
3. Analytic and algebraic theory of systems of equations
4. First order equations
5. Elliptic equations, boundary value problems
6. Hyperbolic equations, Cauchy problem
7. Parabolic equations

8. Mixed equations
9. Classification, characteristics
10. Linear equations of higher order
11. Non-linear equations, special types
12. Eigenvalues, eigenfunctions
13. Approximate methods

Difference Equations, Functional Equations

1. Finite differences and difference equations
2. Generalizations
3. Functional equations

Integral and Integrodifferential Equations

1. Linear integral equations
2. Singular integral equations
3. Integrodifferential equations
4. Non-linear integral equations
5. Special integral equations

Calculus of Variations

1. Theory in the large, topological methods
2. Applications

TOPOLOGICAL ALGEBRAIC STRUCTURES

Topological Lattices

Topological Groups

1. Representations
2. Groups from geometry
3. Semigroups and other generalizations

Lie Groups and Algebras

1. Lie groups
2. Representations
3. Lie Algebras, Lie rings

Topological Rings

Topological Vector Spaces, Functional Analysis

1. Function spaces: general
2. Topological vector spaces
3. Ordered vector spaces
4. Banach spaces
5. Hilbert spaces
6. Special function spaces
7. Distributions
8. Linear operators
9. Groups and semi-groups of linear operators
10. Non-linear operators
11. Rings of operators, group algebras, abstract topological algebras and their representations
12. Applications of functional analysis; analysis of differential and integral operators

TOPOLOGY

Topology: General

1. Sets on a line and in Euclidean space
2. Covering theorems
3. Foundations, topological spaces, abstract theory, limits and generalized limits
4. Metric and uniform spaces
5. Topology of point sets, curves, continua
6. Fixed point properties
7. Topological dynamics
8. Applications to analysis

Topology: Algebraic

1. Homology and cohomology
2. Homotopy
3. Fibre bundles
4. Manifolds
5. Fixed point theorems
6. Links, knots
7. Complexes and polyhedra
8. Topology of group spaces and H-spaces
9. Transformations and special mappings
10. Dimension theory
11. Graphs, four color problem

GEOMETRY

Geometries, Euclidean and other

1. Foundations
2. Elementary geometry
3. Triangles, tetrahedra
4. Circles, spheres, inversive geometry
5. Constructions
6. Finite geometries, configurations, regular figures, divisions of space
7. Vectors, quaternions, tensor algebra
8. Coordinates, analytic methods
9. Conics, quadric surfaces
10. Affine geometry
11. Projective geometry
12. Non-Euclidean geometry
13. n-dimensional and hypercomplex geometries
14. Minkowski geometry
15. Descriptive geometry

Convex Domains, Distance Geometries

1. Convex regions, Brunn-Minkowski theory
2. Extremum properties and geometric inequalities
3. Distance geometries

Differential Geometry

1. Direct methods
2. Classical differential geometry
3. Vector and tensor analysis
4. Kinematic methods and integral geometry
5. Minimal surfaces
6. Families of curves, nets, webs
7. Deformation of surfaces
8. Differential line geometry Laguerre and other sphere geometries
9. Lineal and higher order elements
10. Differential geometry in the large
11. Projective differential geometry
12. Differential geometry under other groups: affine, inversive, conformal, non-Euclidean

Manifolds, Connections

1. Rieumannian geometry
2. Paths and connections: general
3. Non-Riemannian geometry, conformal, affine, projective connections
4. Finsler spaces, abstract differential geometry

Complex Manifolds

Algebraic Geometry

1. Special varieties, curves, surfaces
2. General theory of varieties, surfaces
3. General theory of curves
4. Intersection theory
5. Group varieties, Abelian, equivalence theories
6. Algebraic transformations
7. Algebraic functions

NUMERICAL ANALYSIS

Numerical Methods

1. General mathematical methods, iteration
2. Monte Carlo methods
3. Interpolation, smoothing, least squares, curve fitting, approximation of functions
4. Computation of special functions, series, integrals
5. Linear inequalities, linear programming
6. Linear equations, determinants, matrices
7. Eigenvalues, eigenvectors, Rayleigh-Ritz method
8. Non-linear systems
9. Roots of algebraic and transcendental equations
10. Numerical differentiation and integration, mechanical quadrature

11. Ordinary differential equations
12. Partial differential equations
13. Difference and functional equations
14. Integral and integro-differential equations
15. Error analysis
16. Graphical methods, nomography
17. Harmonic analysis and synthesis
18. Tables

Computing Machines

1. Digital computers: logic and design
2. Digital computers: coding and programming
3. Analogue computers
4. Results of computation by machine

PROBABILITY

1. Foundations
2. Elementary theory
3. Distributions
4. Limit theorems
5. Stochastic processes: general theory
6. Markov processes
7. Stationary processes
8. Special processes, random walks
9. Applications

STATISTICS

1. Elementary descriptive statistics
2. Graduation
3. Distributions of statistical functions
4. Estimation theory (parametric case)
5. Testing of hypotheses (parametric case)
6. Non-parametric methods and order statistics
7. Design and analysis of experiments
8. Decision theory
9. Multistage decision procedures, sequential analysis
10. Statistical engineering, quality control
11. Sampling surveys
12. Time series
13. Applications

PHYSICAL APPLICATIONS

Mechanics of Particles and Systems

1. Foundations
2. Statics
3. Kinematics, mechanics, linkages
4. Dynamics
5. Oscillations, stability
6. Exterior ballistics, artificial satellites
7. Variable mass, rockets

Statistical Thermodynamics and Mechanics

1. Gases
2. Liquids
3. Solids, crystals
4. Quantum statistical mechanics
5. Statistical thermodynamics
6. Irreversible thermodynamics

Elasticity, Plasticity

1. Foundations of mechanics of deformable solids
2. Plane stress and strain
3. Three-dimensional problems
4. Torsion and bending
5. Beams and rods
6. Plates, shells and membranes
7. Anisotropic bodies
8. Vibrations, structural dynamics
9. Stability, buckling, failure
10. Wave propagation
11. Visco-elasticity
12. Plasticity, creep
13. Soil mechanics
14. Thermo-mechanics

Structure of Matter
1. Liquid state
2. Solid state

Fluid Mechanics, Acoustics
1. Foundations
2. Incompressible fluids: general theory
3. Incompressible fluids with special boundaries
4. Free surface flows, water waves, jets, wakes
5. Viscous fluids
6. Boundary layer theory
7. Stability of flow
8. Turbulence
9. Compressible fluids: general theory
10. Compressible fluids: subsonic flow
11. Compressible fluids: transonic flow
12. Compressible fluids: supersonic and hypersonic flow
13. Shock waves
14. Acoustics
15. Non-Newtonian fluids
16. Magneto-hydrodynamics
17. Diffusion, filtration

Optics, Electromagnetic Theory, Circuits
1. Geometric optics
2. Physical optics
3. Electron optics
4. Electromagnetic theory
5. Electro- and magnetostatics
6. Waves and radiation
7. Diffraction, scattering
8. Antennas, wave-guides
9. Circuits, networks
10. Technical applications

Classical Thermodynamics, Heat Transfer
1. Classical thermodynamics
2. Heat and mass transfer
3. Combustion
4. Chemical kinetics

Quantum Mechanics
1. General theory
2. Quantum field theory
3. Atomic and nuclear physics
4. Elementary particles

Relativity
1. Special relativity
2. General relativity
3. Unified field theories
4. Other relativistic theories

Astronomy
1. Celestial mechanics
2. Galactic and stellar dynamics
3. Three and n-body problems
4. Orbits
5. Stellar structure
6. Stellar atmospheres, radiative transfer
7. Hydrodynamic and hydromagnetic problems
8. Cosmology
9. Special problems
10. Radio astronomy

Geophysics
1. Hydrology, hydrography, oceanography
2. Meteorology
3. Seismology
4. Potentials, prospecting, figure of the earth
5. Geo-electricity and magnetism
6. Geodesy, mapping problems

OTHER APPLICATIONS

Economics, Management Science
1. Econometrics
2. Actuarial theory
3. Management science, operations research

Programming, Resource Allocation, Games
1. Linear and non-linear programming, scheduling
2. Games

Biology and Sociology
1. Biology
2. Genetics
3. Demography

4. Sociology
5. Psychology

Information and Communication Theory
1. Information theory
2. Communication theory
3. Linguistics

Control Systems
1. Servomechanisms
2. Switching theory, relays

HISTORY, BIOGRAPHY

1. Ancient and medieval mathematics
2. Modern mathematics
3. India, Far East, Maya
4. Astronomy and physics
5. Biography of
6. Obituary of
7. Collected or selected works of

MISCELLANEOUS

1. General text-books
2. Collections of formulas
3. Bibliography
4. Dictionaries
5. Recreations

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